AMERICAN JOURNAL of PHYSICS

Devoted to the Instructional and Cultural Aspects of Physical Science

VOLUME 21 NUMBER 4

APRIL 1953

Published nine times per year for the

AMERICAN ASSOCIATION OF PHYSICS TEACHERS

by the

AMERICAN INSTITUTE OF PHYSICS

Incorporated

PRINCE & LEMON STS., LANCASTER, PA., OR 57 EAST 55 STREET, NEW YORK 22, N. Y.

QC
1
F331

AAPT Summer Meeting, Pittsburgh, Pennsylvania. See p. 309.

BOSTON UNIVERSITY
COLLEGE OF LIBERAL AR
LIBRARY

AMERICAN ASSOCIATION OF PHYSICS TEACHERS

OFFICERS

PAUL E. KLOPSTEG, *President*
National Science Foundation
Washington, D. C.

MARSH W. WHITE, *President-Elect*
Pennsylvania State College
State College, Pennsylvania

R. F. PATON, *Secretary*
University of Illinois
Urbana, Illinois

FRANCIS W. SEARS, *Treasurer*
Massachusetts Institute of Technology
Cambridge, Massachusetts

EXECUTIVE COMMITTEE

V. E. EATON
W. C. MICHELS
WILL V. NORRIS
W. S. WEBB
M. W. ZEMANSKY

The officers and editor are also members of the executive committee.

COUNCIL

D. M. BENNETT
WILLARD GEER
C. A. HODGES
G. E. C. KAUFFMANN
R. W. LEFLER
W. J. MICHELS
R. R. PALMER
W. B. PIETENPOL
R. L. PRICE
O. L. RAILSBACK
F. VERBRUGGE
C. WILLIAMSON

The members of the executive committee are also members of the council.

AMERICAN JOURNAL OF PHYSICS

Editor
THOMAS H. OSGOOD, *Michigan State College*

Assistant Editor
B. H. DICKINSON, *Michigan State College*

Associate Editors
DAVID L. FALKOFF, *Brandeis University and Massachusetts Institute of Technology*
IRA M. FREEMAN, *Rutgers University*
A. C. HELMHOLZ, *University of California, Berkeley*
W. W. MCCORMICK, *University of Michigan*
NORA M. MOHLER, *Smith College*
C. W. UFFORD, *University of Pennsylvania*
G. H. VINEYARD, *University of Missouri*
C. N. WALL, *University of Minnesota*
ROBERT W. YOUNG, *U. S. Navy Electronics Laboratory, San Diego*

THE AMERICAN JOURNAL OF PHYSICS is devoted to the instructional and cultural aspects of physical science. Previous to 1940 this publication was known as *The American Physics Teacher*. It is published monthly, except during June, July and August, by the American Institute of Physics for the American Association of Physics Teachers at Prince and Lemon Streets, Lancaster, Pennsylvania.

Manuscripts for publication should be submitted to Dr. Thomas H. Osgood, *American Journal of Physics*, Michigan State College, East Lansing, Michigan. The authors' institutions are requested to pay an optional **publication charge** of $4 per page which, when honored, entitles them to 100 free reprints. Instructions will be sent with galley proofs.

Proof and correspondence concerning papers in the process of being printed should be addressed to the American Institute of Physics, 57 East 55 Street, New York 22, New York.

Subscription Price

United States and Canada $6.00
Elsewhere 7.00

Back Numbers

Complete set: Vol. 1, 1933–Vol. 20, 1952—$140.00
Yearly back number rate: $8.00
Single copies: $2.00 each 1933–1936; $1.50 each thereafter.

Subscriptions, renewals, and orders for back numbers should be addressed to the American Institute of Physics, 57 East 55 Street, New York 22, New York. Members and junior members of the American Association of Physics Teachers receive the *American Journal of Physics*.

Changes of address should be addressed to the American Institute of Physics.

The contents of the *American Journal of Physics* can be found indexed in the *Education Index*.

Entered as second-class matter April 5, 1948, at the post office at Lancaster, Pa., under the Act of August 24, 1912.

AMERICAN JOURNAL of PHYSICS

A Journal Devoted to the Instructional and Cultural Aspects of Physical Science

VOLUME 21, NUMBER 4 APRIL, 1953

Construction of a Diffraction Grating Spectrograph

R. L. PURBRICK
Willamette University, Salem, Oregon

(Received August 11, 1952)

The construction and operation of a simple Wadsworth-mounted 21-ft 10-in. diffraction grating spectrograph is described. The instrument covers the range from 9000A to 1800A in the first order with a linear dispersion of 5A/mm.

SPACE for mounting is one of the problems which presents itself to a group planning the construction of a grating spectrograph with sufficient dispersion for both research and class use. After considerable investigation we decided to construct a Wadsworth-type mounting for a 21-ft 10-in. diffraction grating. This mounting is compact, and a normal stigmatic spectrum is produced at the normal to the grating. Various workers have described their versions of this type of mounting.[1] Our modification yields a relatively simple instrument, both from the standpoint of construction and operation.

DESCRIPTION

The arrangement of optical elements for a Wadsworth-type mounting is shown in Fig. 1. Two equations express the conditions which must be satisfied:

$$n\lambda = d(\sin i \pm \sin r), \tag{1}$$

$$s = R/(1+\cos i), \tag{2}$$

where n is the order number, λ the wavelength, d the grating space, i the angle of incidence, r the angle of reflection, s the distance between grating and photographic plate, and R the radius of curvature of the grating. Provision must be made for changing i and s.

Figure 2 shows our version of the mounting. The main support is formed by 6-in. channel irons welded together by $\frac{1}{4}$-in. steel plates. Five 4-in. diameter steel pipes, 3 ft in length and with flanges at both ends, constitute the legs. Another 6-in. channel iron has the grating at one end and the plateholder at the other. A Timken 366 bearing mounted directly beneath the grating supports one end of this beam, and a similar bearing near the plateholder supports the other

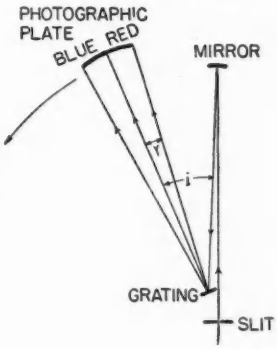


FIG. 1. Arrangement of optical elements for a Wadsworth-type mounting. The arrow indicates the direction the plateholder must be moved for higher wavelengths.

[1] W. F. Meggers and K. Burns, Bur. Standards Sci. Paper 411, **18**, 185 (1922); R. F. Jarrell, J. Opt. Soc. Am. **32**, 666 (1942); J. W. Forrest and H. W. Straat, J. Opt. Soc. Am. **32**, 669 (1942).

end. A steel pipe extends from the slit towards the mirror. This prevents light from the slit from directly striking the photographic plate.

The concave mirror is 6 in. in diameter and has a radius of curvature the same as that of the grating. It renders the light incident on the grating parallel and is the primary factor contributing to a stigmatic image.

The grating was ruled by Dr. John Strong's group at The Johns Hopkins University. It is 6 in. in diameter, has a 21-ft 10-in. radius of curvature, 14 400 lines per inch, and high intensity in the second order. The grating is mounted in a dustproof cover, and provision is made for rotation about three mutually perpendicular axes to facilitate alignment.

A round straight steel bar at the plate end of the instrument serves as a rail for moving this end of the channel iron on which the grating and plateholder are mounted. The rail is straight, but this end of the channel iron follows the arc of a circle when moved; hence it was necessary to mount it as shown in Fig. 3. Three motions are allowed: (1) movement along the rail, (2) rotation of the channel iron about its point of support, and (3) movement along a line connecting the grating and plateholder. This arrangement has one advantage over a steel bar ground to the arc of a circle; the channel iron here can be rotated for any position of the grating on its steel plate support.

The in-and-out motion of the plateholder, which must be carried out to satisfy Eq. (2), is provided for as shown in Fig. 3. Set screws at the ends of the 2 in. ×10 in. photographic plate are used to curve the plate for proper focus.

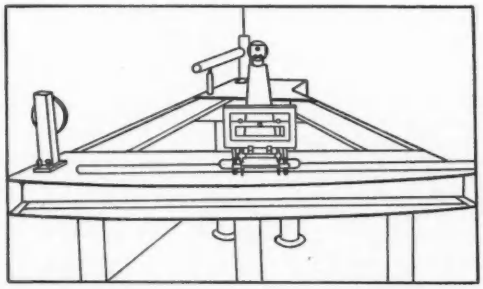


FIG. 2. The spectrograph mounting described in this paper. The case for the instrument is not shown

Unfortunately this curvature varies slightly with spectral region. Focusing can be visual with a ground glass plate, or a scale may be mounted on the channel iron and photographic alignment used to calibrate the scale.

The mounting is encased in a cabinet formed from $\frac{3}{4}$-in. plywood from the top of the legs upward. The case follows the outline of the mounting and is 2 ft 5 in. in height, 8 ft 5 in. wide, and 13 ft long. Two 1 ft ×4 ft hinged doors are located at the plate end.

PERFORMANCE

The instrument is aligned with Eqs. (1) and (2) in mind. Figure 4 illustrates how critical the position of the plateholder is for proper focus.

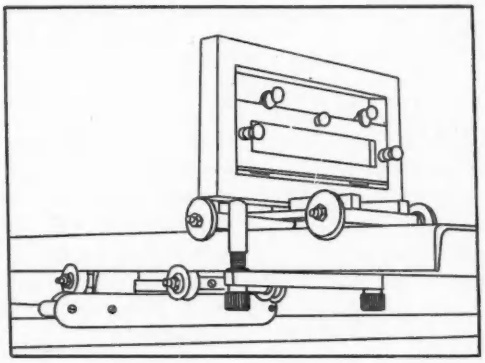


FIG. 3. The plateholder end of the movable beam.

The slit width was held constant; the mercury 2536A line was photographed, and its width was measured for various positions of the plateholder. The curve indicates that the plate should be positioned to millimeter accuracy.

The slit and grating rulings must be parallel. This is best achieved photographically. Two neon lines at 6143A and 6164A are shown in Fig. 5; (A) was taken with a small angle between slit and rulings while in (B) they were parallel and the images much sharper.

The spectrograph is mounted on a concrete floor laid on the ground, and to date no difficulties have been encountered as a result of vibration. During long exposures an effort is

made to keep the room temperature reasonably constant. However, this has posed no serious problem in exposures up to 12 hours.

The instrument covers the region from 9000A to 1800A in the first order with a linear dispersion of approximately 5A/mm. The dispersion varies about 0.2A/mm between the red and ultraviolet.

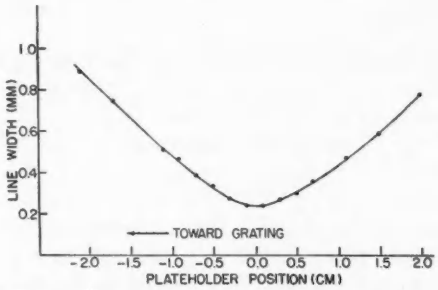
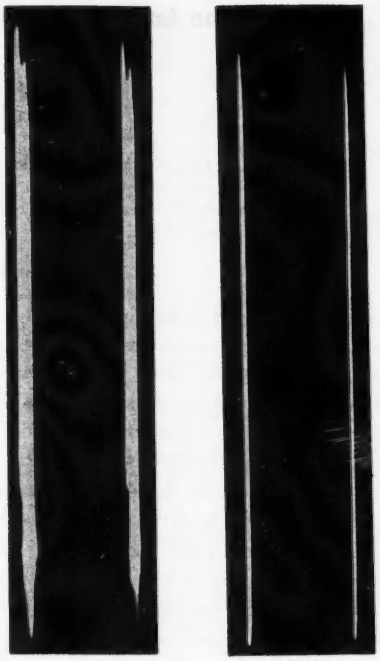


FIG. 4. The variation of spectral line width as a function of plateholder position on the moving beam. Negative values indicate displacement from the point of sharpest focus toward the grating.



(a) (b)

FIG. 5. Photographs of two neon lines showing the necessity of having slit and grating rulings parallel. (a) was taken with a small angle between slit and grating rulings; in (b) they are parallel.

The author is indebted to the Research Corporation for a Frederick Gardner Cottrell grant, which aided in this project, and to students Robert Bennett, James Carpenter, Lawrence Cherry, Alan Goudy, Frederick Gruver, and Philip Phipps, who helped in the construction as part of a special problems course. James Carpenter, Sr., contributed greatly to the construction of the welded portion of the instrument.

The chief objects of the [Royal] Institution were to provide a speedy and general diffusion of knowledge of all new and useful improvements, to teach the application of scientific discoveries to the improvement of arts and manufactures in England, and increase domestic comfort and convenience. Under Rumford's guiding hand a house was purchased in Albemarle Street and laid out according to his plans. There was a lecture theatre, a model room to exhibit mechanical inventions and improvements, workshops and kitchens. A school for mechanics was established to extend the benefits of science to the working poor. Professors and lecturers were appointed to give the nobility and wealthy an opportunity to share in the advantages of scientific instruction. Various committees were established to supervise scientific investigation; a printing press was bought, and a journal was to be published. The Institution was to be a "clearing-house" of scientific information, a source from which great good could flow. As Rumford said: "I am only desirous that science and art should once be brought cordially to embrace each other, and to direct their united efforts to the improvement of agriculture, manufactures, and commerce, and to the increase of domestic comfort."
—GEORGE A. FOOTE, Isis 43, 7 (1952).

On a Routine Analytic Method for the Solution of Problems in Statics

A. W. Simon
Alabama Polytechnic Institute, Auburn, Alabama
(Received September 15, 1952)

A routine analytic method for the solution of problems in statics is given. The method requires no geometry other than that required for the identification of the coordinates of certain cardinal points involved in the problem. The method is illustrated by the detailed solution of an actual case.

THE methods of solution of problems in statics given in most textbooks are characterized by the fact that they depend largely on the use of special tricks and artifices, which are applicable in one case but not in another, with the result that the student is usually at a loss how to proceed and often wastes a great deal of time before he does find the proper method of attack. Furthermore, the methods of solution given usually make the mechanics subservient to the mathematics, rather than vice versa. In this paper a method is presented, which is straightforward and directly applicable to *any* problem in statics, and in which the equations set up on the basis of the mechanics involved serve to indicate the best method of mathematical solution. Furthermore, the method to be presented requires no geometry (or trigonometry) beyond the ability to identify the coordinates of points, the entire solution being analytic, that is algebraic, after the coordinates of certain cardinal points involved in the problem have been determined.

I. THEORETICAL BASIS

A method for the solution of problems in statics, and one particularly convenient in the case of forces in space, can be based on the analytic forms of the fundamental equations

$$\sum X=0, \quad (1) \qquad \sum M_x=0, \quad (4)$$

$$\sum Y=0, \quad (2) \qquad \sum M_y=0, \quad (5)$$

$$\sum Z=0, \quad (3) \qquad \sum M_z=0, \quad (6)$$

in which, as usual, X, Y, Z represent the axial components of the force $\mathbf{F}$, and M_x, M_y, M_z, represent the axial components of the moment, with respect to the origin, of the force $\mathbf{F}$.

In *analytic* form, Eqs. (1), (2), and (3) become

$$\sum F(x_2-x_1)/r=0, \quad (7)$$

$$\sum F(y_2-y_1)/r=0, \quad (8)$$

$$\sum F(z_2-z_1)/r=0, \quad (9)$$

where x_1, y_1, z_1, and x_2, y_2, z_2, represent the coordinates of an (arbitrarily selected) first point P_1 and second point P_2, respectively, on the line of action of the force $\mathbf{F}$, and r represents the distance between the two points. The distance r is, as usual, given by

$$r^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2. \quad (10)$$

Furthermore, in *analytic* form, Eqs. (4), (5), and (6) become

$$\sum(yZ-zY)=0, \quad (11)$$

$$\sum(zX-xZ)=0, \quad (12)$$

$$\sum(xY-yX)=0, \quad (13)$$

in which x, y, z, on the right-hand side now represent the coordinates of a point on the line of action of the force $\mathbf{F}$. For convenience this point may be taken to coincide with either one of the points 1 and 2, the coordinates of which have previously been determined.

In the following we shall write the coordinates of the points P_1 and P_2 in the form $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, and the vector $\bar{P}_1\bar{P}_2$ extending from P_1 to P_2 as

$$\bar{P}_1\bar{P}_2=[(x_2-x_1), (y_2-y_1), (z_2-z_1)].$$

While any point may be chosen as the origin of coordinates, the moment equations can ordinarily be simplified if a point of concurrency of two or more forces is selected as the origin, provided, of course, such a point exists, as is usually the case. Furthermore, if part of the data is in

trigonometric form, a convenient artifice is to choose two points which are unit distance apart. The direction cosines of the corresponding angles may then be used directly.

With regard to the direction of the unknown forces, an alternative of the above scheme is to assume all unknown forces to be in such directions as to correspond to tensions; a positive result then indicates tension, a negative result a compression.

II. PRACTICAL PROCEDURE

(1) Identify the coordinates of a first and a second point on the line of action of each force involved, both known and unknown.

(2) Compute in order for each force the differences (x_2-x_1), (y_2-y_1), and (z_2-z_1), the magnitude of r, and the ratios $(x_2-x_1)/r$, $(y_2-y_1)/r$, and $(z_2-z_1)/r$; that is, the direction cosines of the vector $\bar{P}_1\bar{P}_2$, or, alternatively, the components of a unit vector in the direction of $\bar{P}_1\bar{P}_2$.

(3) Insert the values so found in Eqs. (7), (8), and (9).

(4) Insert the same direction cosines and the coordinates of either one of the points P_1 and P_2 (or, if desired, other convenient point on the line of action of the force) in Eqs. (11), (12), and (13).

Solve the resulting six equations simultaneously to determine the magnitude of the unknown forces. A positive result means the corresponding force is actually in the direction from P_1 to P_2, while a negative result means it is opposite thereto.

III. ILLUSTRATIVE EXAMPLE

The method can best be illustrated by working out in detail an actual case.

For this purpose consider the structure pictured in Fig. 1, in which a horizontal boom is supported by a ball-and-socket joint, or equivalent, at one end and by two cables at the other end, while a force of 100 lb is applied in the direction shown. Let the dimensioning be that of the figure. It is desired to determine the force which each of the cables exerts on the boom, as well as the three components of the reaction at the ball-and-socket joint. If the various cardinal points are lettered as in Fig. 1, the desired forces are those along (but not necessarily in the same direction as) the vectors $\bar{A}\bar{B}$, $\bar{A}\bar{C}$, $\bar{A}\bar{D}$, $\bar{D}\bar{E}$, and

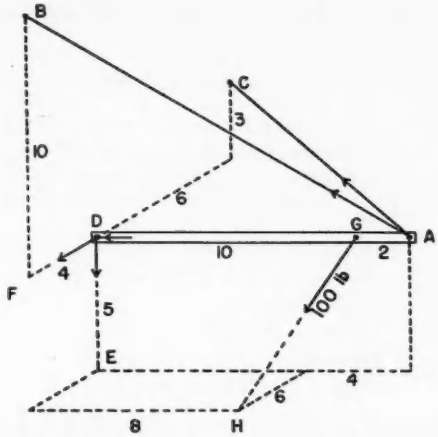


FIG. 1. System of forces analyzed in the illustrative example. The directions of the forces indicated are arbitrarily assumed.

$\bar{D}\bar{F}$, with the known force in the direction of the vector $\bar{G}\bar{H}$.

If now the origin is taken at A and a right-handed system of coordinates with the x axis extending horizontally to the right and the y axis vertically upward is employed, the coordinates of the cardinal points involved are

$A(0, 0, 0)$, $B(-12, 10, 4)$, $C(-12, 3, -6)$,
$D(-12, 0, 0)$, $E(-12, -5, 0)$, $F(-12, 0, 4)$,
$G(-2, 0, 0)$, $H(-4, -5, 6)$.

It is to be noted that further reference to the figure is now unnecessary, the rest of the computation following automatically.

For each of the vectors involved we now write the two end points in inverted order and subtract corresponding coordinates. Thus, for $\bar{A}\bar{B}$, we write

$$\begin{array}{l} B(-12, 10, 4) \\ A(\ \ 0,\ \ 0, 0) \\ \hline \bar{A}\bar{B}=[-12, 10, 4]. \end{array}$$

The corresponding value of r is

$$r=\sqrt{260}=16.1,$$

while the corresponding direction cosines or components of a unit vector in the direction of $\bar{A}\bar{B}$ are

$$-12/16.1=-0.744, \quad 10/16.1=0.620, \quad 4/16.1=0.248.$$

The results may be conveniently written in the form

$$\bar{A}\bar{B}_1=[-0.744, 0.620, 0.248],$$

where the subscript 1 indicates a unit vector.

If these computations are carried out for each of the vectors involved, and the results gathered in a table, we have

$$\begin{aligned}
\bar{A}\bar{B}_1&=[-0.744, 0.620, 0.248],\\
\bar{A}\bar{C}_1&=[-0.873, 0.218, -0.436],\\
\bar{A}\bar{D}_1&=[-1, 0, 0],\\
\bar{D}\bar{E}_1&=[0, -1, 0],\\
\bar{D}\bar{F}_1&=[0, 0, 1],\\
\bar{G}\bar{H}_1&=[-0.248, -0.620, 0.744].
\end{aligned}$$

In the case of such forces as AD, DE, and DF, which are in directions parallel to the coordinate axes, the corresponding direction cosines can, of course, be determined by inspection without recourse to calculation.

Substitution in Eqs. (7) to (9), inclusive, yields (the two letter symbols now represent the magnitudes of the corresponding forces)

$$-0.744AB-0.873AC-AD-0.248\times 100=0, \quad (7a)$$

$$0.620AB+0.218AC-DE-0.620\times 100=0, \quad (8a)$$

$$0.248AB-0.436AC+DF+0.744\times 100=0. \quad (9a)$$

We turn next to the moment equations. Selecting first Eq. (11), we note that if we choose the points A, D, and G, for each of which $y=z=0$, the corresponding equation reduces to the form $0=0$. This is merely the mathematical expression of the fact, obvious from the figure, that none of the forces involved have a moment around the chosen x axis.

Turning next to Eq. (12), we note that if for the forces AB, AC, and AD we select the point A, for which $z=x=0$, the corresponding terms vanish; while, if for the force DF we select the point D for which $z=0$, the only term left is that corresponding to $-xZ$, which becomes $-(-12)DF$; further, if for the force DE we select again the point D, the first term vanishes for the same reason as before, while the second term vanishes because for this force $Z=0$; hence DE makes no contribution. This leaves finally the known force GH, for which we select the point G, and have as its only contribution the term corresponding to $-xZ$, which is $-(-2)(0.744)100$. Hence, Eq. (12) reduces to

$$-(-12)(1)DF-(-2)(0.744)100=0. \quad (12a)$$

Similarly, by a judicious choice of points involved in Eq. (13), the latter reduces to

$$-12(-1)DE+(-2)(-0.620)100=0. \quad (13a)$$

A glance at Eqs. (7a) to (13a), inclusive, now indicates the method to be followed in completing the algebraic part of the solution.

The solution is $DE=-10.3$, $DF=-12.4$, $AB=27.9$, $AC=158$, $AD=-184$. The minus signs indicate that the corresponding forces are opposite to the directions originally assumed.

It will be noted that an alternative choice of origin is the point D, in which case (axes parallel to the original directions) the coordinates of the points A and G become

$$A(12, 0, 0) \quad \text{and} \quad G(10, 0, 0),$$

and the corresponding moment equations reduce to

$$-12(0.248)AB-12(-0.436)AC-10\times 100\times 0.744=0,$$

$$+12(0.620)AB+12(0.218)AC+10\times 100(-0.620)=0.$$

The direction cosines are, of course, unchanged by a shift of origin provided the axes of the new coordinate system are taken parallel to and in the same direction as the original.

The Constant Battery Wheatstone Bridge

T. Townsend Smith
University of Nebraska, Lincoln, Nebraska
(Received July 31, 1952)

The article discusses the sensitivity of a constant battery Wheatstone bridge and shows that under certain conditions the bridge sensitivity will be independent of the magnitude of the resistance to be measured. A chart is given, which indicates for a given resistance the bridge ratio which provides the highest sensitivity.

The bridge is insensitive for measuring resistances which are large in comparison with the resistance of the ratio arms.

THE question of the sensitivity of a constant battery Wheatstone bridge is interesting and of some importance in an electrical laboratory.

Suppose that a battery of emf E and of negligible resistance is connected to a Wheatstone bridge as indicated in Fig. 1. Suppose further that the bridge as shown in Fig. 1 is in balance and that the resistance R_1 is then increased to $R_1+\Delta R_1$. The sensitivity S of the bridge may be defined in terms of the galvanometer deflection D and the relative change in R_1, as

$$S=D(R_1/\Delta R_1). \tag{1}$$

The sensitivity thus defined can be expressed in terms of E, of the resistances of the various arms of the bridge, and of the resistance G and the sensitivity σ of the galvanometer, where σ is, by definition,

$$\sigma=D/I_G. \tag{2}$$

As a first step one may write for the current through the galvanometer, in terms of the current through R_1,

$$I_G=\frac{I_1(\Delta R_1)}{G(1+R_2/R_4)+R_1+R_2}. \tag{3}$$

One may now express I_1, to a first approximation, neglecting terms in I_G/I_1, since I_G/I_1 will be small for a bridge only a little out of balance. To this approximation the value of I_1 may be written

$$I_1=E/(R_1+R_3). \tag{4}$$

Equations (1), (2), (3), and (4) may be combined to give an expression for the sensitivity of the bridge. Thus,

$$S=D(R_1/\Delta R_1)=E\sigma\frac{R_1}{R_1+R_3}\cdot\frac{1}{G(1+R_2/R_4)+R_1+R_2}. \tag{5}$$

One may express S as a function of a parameter $(R_1+R_2)/R$, where R is used as a notation for R_3+R_4, with the aid of the relations

$$R_1/R_3=R_2/R_4=(R_1+R_2)/R. \tag{6}$$

So expressed, Eq. (5) becomes

$$S=E\sigma\frac{(R_1+R_2)/R}{1+(R_1+R_2)/R}\cdot\frac{1}{G[1+(R_1+R_2)/R]+R(R_1+R_2)/R}. \tag{7}$$

If ρ is used as a symbol for $(R_1+R_2)/R$, Eq. (7) becomes

$$S=E\sigma\frac{\rho}{(1+\rho)}\cdot\frac{1}{G(1+\rho)+R\rho}. \tag{8}$$

Equations (7) or (8) show that the sensitivity of the bridge does not depend upon R_1 directly but only on the ratio $(R_1+R_2)/R$. As long as this ratio is kept constant, the sensitivity of the

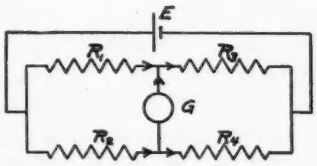


Fig. 1. A "center-connected" Wheatstone bridge, with the battery connected between the junction of the ratio arms (R_3-R_4) and the junction of the known and unknown resistances (R_1-R_2).

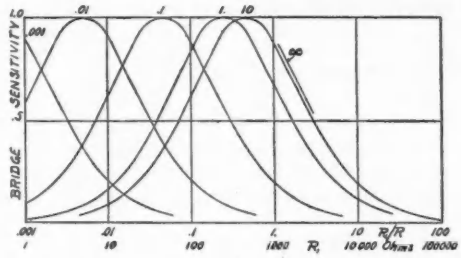


FIG. 2. The sensitivity of a constant battery, center-connected Wheatstone bridge. This chart may be used to obtain the resistance ratio which should be used to get high sensitivity in measuring a particular resistance. The ordinates are S/S_{max}. The abscissas are marked in terms of R_1/R, and also in terms of R_1, for the case $R=1000$ ohms.

bridge will be unchanged and any resistance may be measured with the same precision, in so far as the precision depends on the sensitivity of the bridge.

The preceding paragraph should not be interpreted as asserting that with this type of bridge all resistances can be measured with the same precision, for that is not true, though it is true that all resistances of less than a certain critical value (which depends on R and G) may be measured with the same precision. If R_1 is greater than this critical value, the sensitivity is less. If R_1/R is large, the bridge is very insensitive.

The bridge sensitivity is a function of ρ and G. It is a maximum when ρ is $\frac{1}{2}$ and when G is $\frac{1}{3}R$. To find the value of ρ for which the sensitivity is a maximum, G remaining constant, it is convenient to write Eq. (8) in the form

$$S=\frac{E\sigma}{[G(1+\rho)^2/\rho]+R(1+\rho)}, \tag{9}$$

and to find the value of ρ for which the denominator is stationary. This value is given by

$$\rho^2=G/(R+G). \tag{10}$$

To find the value of G which makes S a maximum, one needs first to express the galvanometer sensitivity σ in terms of the galvanometer resistance G. For a given type of galvanometer this relation is expressed to a fair approximation by the equation[1]

$$\sigma=\sigma_0 G^{\frac{1}{2}}. \tag{11}$$

[1] T. T. Smith, *Electricity for Students of Science and Engineering* (International Textbook Company, Scranton, 1949), p. 128.

By combining Eqs. (9), (10), and (11), one gets an expression for S:

$$S=E\sigma_0\frac{G^{\frac{1}{2}}}{[G^{\frac{1}{2}}+(G+R)^{\frac{1}{2}}]^2}. \tag{12}$$

The value of G for which S is a maximum is found by setting equal to zero the derivative of the expression on the right side of Eq. (12). This procedure gives

$$G=\tfrac{1}{3}R. \tag{13}$$

If now this value for G is inserted in Eq. (10), the result is

$$\rho=(R_1+R_2)/R=\tfrac{1}{2}. \tag{14}$$

Thus, as indicated above, the bridge is most sensitive when ρ is $\frac{1}{2}$ and when G is $\frac{1}{3}R$.

Since R_1 and R_2 are both positive, it follows from Eq. (14) that the largest resistance which can be measured with the maximum sensitivity is $\frac{1}{2}R$. An expression for this maximum sensitivity can be obtained from Eq. (9) by substituting $\frac{1}{3}R$ for G and $\frac{1}{2}$ for ρ. The result is

$$S_{max}=E\sigma/3R. \tag{15}$$

The curves of Fig. 2 show the ratio S/S_{max} as a function of the bridge ratio ($r=R_1/R_2=R_3/R_4$) and of the ratio R_1/R. The formula by which these curves are calculated is obtained by expressing ρ in terms of R_1/R and r, substituting this value in Eq. (9), and then combining the resulting equation with Eq. (15). The expression for ρ is

$$\rho=\frac{R_1+R_2}{R}=\frac{R_1}{R}\left(1+\frac{1}{r}\right)=\frac{R_1}{R}\left(\frac{1+r}{r}\right), \tag{16}$$

and the resulting formula for calculating S/S_{max} is

$$\frac{S}{S_{max}}=\frac{9}{5+4\left(\frac{1+r}{r}\right)\frac{R_1}{R}+\left(\frac{r}{r+1}\right)\frac{R}{R_1}}. \tag{17}$$

The curves are drawn for various values of r from 0.001 to 10, with an indication of the position of the curve for the limiting bridge ratio of infinity.

We have in our electrical laboratory a bridge of the type here discussed with a resistance R of 1000 ohms and with a dry cell for E. The galvanometer has a sensitivity of 45×10^6 mm/am-

pere and a resistance of 265 ohms. The peak sensitivity is about 2600, which is almost exactly the value which would be given by Eqs. (9) and (10), or by Eqs. (15) and (11). A chart showing the curves of Fig. 2 hangs by the bridge. As soon as the approximate value of an unknown resistance is found, a glance at the chart enables one to choose the bridge ratio for the highest sensitivity.

Some comments seem to be pertinent:

(1) A change in the galvanometer resistance G shifts the curves of Fig. 2 without greatly altering their form. Take, as an example, a bridge ratio of 0.1. With the best galvanometer ($G=\frac{1}{3}R=333$ ohms for an R of 1000 ohms) the peak sensitivity is at 45 ohms. In our bridge G is 265 ohms, for which the peak sensitivity is at 41 ohms with a ρ value of 0.46 [Eq. (10)]. When $G=500$ ohms the peak sensitivity would be at 52 ohms with a ρ value of 0.58, and for $G=1000$ ohms the peak would be at 71 ohms with a ρ value of 0.71.

(2) A switch for interchanging galvanometer and battery would add little to the usefulness of the bridge. If the battery and galvanometer were interchanged so that the battery would be across the ends of the ratio arms, the sensitivity with a 1-1 bridge ratio would be somewhat increased for small values of R_1. With ratios of 0.1 or 10 the change would decrease the sensitivity for small values of R_1. For large values of R_1 both bridges are insensitive and there is little to choose between the two arrangements, though the end-connected bridge may be made a little more sensitive. For general use, considering both sensitivity and convenience, the center-connected bridge is to be preferred.

(3) This note deals only with a Wheatstone bridge used as a testing set with a constant battery, perhaps a single dry cell. For resistance comparisons of high precision one would normally increase the sensitivity by using an electromotive force that provided currents close to the tolerance limits of the bridge conductors. For such work the conditions for high sensitivity and for best galvanometer resistance are entirely different from those discussed here.[2] For example, for high precision measurements a desirable bridge arrangement is the equal arm bridge, $R_1=R_2=R_3=R_4$, and in this case the best galvanometer resistance is $R_1(G=R_1)$. In such work bridge sensitivities of the order of a million may be obtained with fairly insensitive galvanometers, if R_1 is large enough to use a Wheatstone instead of a Kelvin bridge. With precision measurements the sensitivity should be independent of R_1, if R_1 is large enough for Eq. (11) to be applicable. With a constant battery the sensitivity is proportional to $(1/R_1)^{\frac{1}{2}}$ for large values of R_1.

(4) A bridge of the sort here discussed is not suitable for large bridge ratios. For example, assume that R is 1000 ohms, R_1/R_2 is 0.001, R_1 is about 1 ohm, and a dry cell is applied directly to the junctions R_1-R_2 and R_3-R_4. With these values the power dissipation in R_1 and R_3 will be somewhat over a watt each. This power may be above the tolerance limits of the coils.

(5) For a galvanometer of a given type the sensitivity of the bridge changes very little for relatively large changes in G. For the center-connected bridge the best value of G is $\frac{1}{3}R$. If G is changed by a factor of 3, to R or to $\frac{1}{9}R$, Eq. (12) shows that S changes by a factor of 0.9. Galvanometer resistance G would need to be changed by a factor of nearly 10 to reduce the sensitivity to $\frac{2}{3}$ of the peak value.

[2] T. T. Smith, *Electricity for Students of Science and Engineering* (International Textbook Company, Scranton, 1949), pp. 54–56; Glazebrook's *Dictionary of Applied Physics* (Macmillan Company, New York, 1922), Vol. 2. pp. 713, 714. Other approaches to this question are given in Page and Adams, *Principles of Electricity* (D. Van Nostrand Company, Inc., New York, 1931), pp. 175; Harnwell, *Principles of Electricity and Electromagnetism* (McGraw-Hill Book Company, Inc., New York, 1949), second edition, pp. 133, 134.

Wisconsin Section

The Wisconsin Section of the American Association of Physics Teachers will meet at Wisconsin State College, La Crosse, Wisconsin, on April 24 and 25, 1953.

Spin Waves in Ferromagnetic and Antiferromagnetic Materials*

F. KEFFER,† H. KAPLAN,‡ AND Y. YAFET§
University of California, Berkeley, California
(Received September 22, 1952)

A brief outline is given of the spin-wave approximation to the near ground states of ferromagnetism and antiferromagnetism. Simple pictorial models of spin waves are introduced. These models clarify the striking difference between ferromagnetic spin waves, which obey the dispersion law $\omega \propto k^2$, and antiferromagnetic spin waves, which obey the law $\omega \propto k$. The proper normal modes for antiferromagnetic spin waves are presented. The nature of ferromagnetic and antiferromagnetic resonance is illuminated from a spin-wave point of view.

I. INTRODUCTION

OUR objective in this paper is to give a simple pictorial representation which displays the striking difference between ferromagnetic and antiferromagnetic spin waves. Since the theory of spin waves is not too widely known, however, we shall begin with a few words about this approximation to the nearly ordered, low temperature states of ferromagnetism and antiferromagnetism.[1]

A. Some Remarks about Ferromagnetic Spin Waves

At each lattice site in a ferromagnet let us assume a localized spin with spin quantum number S and with z component of angular momentum $\hbar S_i^z$. In the ground state of the spin system, which will be realized at 0°K, each spin has the maximum allowable value of S_i^z, namely, S. A spin wave may now be described as a sinusoidal disturbance of the spin system with amplitude at each lattice site proportional to $S-S_i^z$.

It was first shown by Bloch[2] that the states near the ground state of a ferromagnet can be approximated by superpositions of these sinusoidal spin waves. We shall indicate the direction of Bloch's quantum-mechanical proof and then go on to a semiclassical approach, introduced by Heller and Kramers,[3] which allows simple pictorial interpretations of spin waves.

Quantum-Mechanical Approach

The exchange Hamiltonian of a system of spins is

$$\mathcal{H} = -2J \sum_{(i,j)} \mathbf{S}_i \cdot \mathbf{S}_j + \text{const}, \qquad (1)$$

where the summation is taken over all combinations of nearest neighbors; J is the exchange integral between nearest neighbors. We shall limit our discussion to spin $S=\frac{1}{2}$, and shall introduce the Pauli matrices $\sigma_i^{\pm}=\sigma_i^x \pm i\sigma_i^y$; σ_i^z. The eigenfunctions of σ_i^z are α_i (eigenvalue $+1$) and β_i (eigenvalue -1). We assume either a small applied field or some anisotropy in order that z be the preferred axis; but for simplicity we will omit such terms from the calculation. The Hamiltonian (1) becomes (apart from the constant)

$$\mathcal{H}' = -\tfrac{1}{2}J \sum_{(i,j)} (\tfrac{1}{2}\sigma_i^+\sigma_j^- + \tfrac{1}{2}\sigma_i^-\sigma_j^+ + \sigma_i^z\sigma_j^z), \qquad (1')$$

which has the ground-state eigenfunction, if N spins are present,

$$\psi_0 = \alpha_1\alpha_2\alpha_3 \cdots \alpha_N. \qquad (2)$$

This is the state at 0°K; it represents the maximum alignment of spins, or saturation magnetization. As the temperature increases the system will be excited out of the ground state. The next state may be thought of as one in which

* This research was assisted in part by the U. S. Office of Naval Research.
† Now at Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania.
‡ Now with Solid State Group, Massachusetts Institute of Technology, Cambridge, Massachusetts.
§ Now at Department of Physics, University of Illinois, Urbana, Illinois.
[1] For an excellent introductory review of the quantum theory of ferromagnetism the reader is referred to J. H. Van Vleck, Revs. Modern Phys. 17, 27 (1945); for an excellent review of antiferromagnetism to J. H. Van Vleck, J. phys. et radium 12, 262 (1951).
[2] F. Bloch, Z. Physik 61, 206 (1930).
[3] G. Heller and H. A. Kramers, Proc. Roy. Acad. Sci. (Amsterdam) 37, 378 (1934). For a simplified reformulation of the semiclassical theory see C. Herring and C. Kittel, Phys. Rev. 81, 869 (1951), especially Appendix A.

a single spin is reversed; however, the function

$$\varphi_j = \alpha_1\alpha_2\cdots\alpha_{j-1}\beta_j\alpha_{j+1}\cdots\alpha_N, \tag{3}$$

is not an eigenfunction of Eq. (1′). An eigenfunction ψ_k may be formed by taking a linear combination of φ_j, each member of the combination containing a reversal at a different lattice site:

$$\psi_k = \sum_j c_j^k \varphi_j. \tag{4}$$

It can be shown that each such proper combination is equivalent to a spin wave of wave number k, and that the allowed values of k can be determined from periodic boundary conditions.[4]

We shall demonstrate the wave-like properties of the solutions (4) for the case of a linear chain. Requiring periodic boundary conditions is equivalent to bending the chain around into a ring so that the first spin is also the $N+1$ spin. Each spin j has two nearest neighbors, $j+1$ and $j-1$, with which it interacts through the exchange integral J. The Schrödinger equation,

$$\mathcal{H}'\psi_k = E\psi_k,$$

allows us to equate the inner product

$$(\varphi_{j'}|\mathcal{H}|\psi_k) = -J[c_{j'-1}^k + c_{j'+1}^k + \tfrac{1}{2}(N-4)c_{j'}^k]$$

with the inner product

$$(\varphi_{j'}|E|\psi_k) = Ec_{j'}^k.$$

Therefore, dropping primes,

$$[E + \tfrac{1}{2}(N-4)J]c_j^k + J(c_{j+1}^k + c_{j-1}^k) = 0. \tag{5}$$

This equation has solutions of the form

$$c_j^k = N^{-\frac{1}{2}}e^{ikja}, \tag{6}$$

where a is the distance between spins. The periodic boundary conditions require $c_j^k = c_{j+N}^k$, and hence, the allowed values of k are given by

$$kaN/2\pi = 0, \pm 1, \pm 2, \cdots, \pm\tfrac{1}{2}(N-1), \pm\tfrac{1}{2}N.$$

The wave-like property of the solutions (4) is shown by Eq. (6). On inserting Eq. (6) into Eq. (5), we obtain an equation relating the energy E and the wave number k:

$$E + \tfrac{1}{2}(N-4)J + 2J\cos ka = 0.$$

For small values of k this reduces to

$$E - E_0 \cong Jk^2a^2,$$

where $E_0 = -\frac{1}{2}NJ$ is the energy of the ground state. It is seen that $E - E_0$ is the energy of the spin wave, E_k. If we quantize this energy, we arrive at the dispersion law

$$E_k = \hbar\omega_k \cong (\text{const})Ja^2k^2 \quad \text{(ferromagnet)}. \tag{7}$$

For a linear chain with $S = \frac{1}{2}$ the value of the constant is unity; for a three-dimensional lattice of cubic symmetry and with arbitrary S the constant is $ZS/3$, where Z is the number of nearest neighbors. The extension of Bloch's theory to arbitrary S was made by Møller.[5]

Semiclassical Approach

It is well known that the classical vector model of a precessing spin gives results in agreement with quantum mechanics if the length of the vector, in units $\hbar$, is taken to be $[S(S+1)]^{\frac{1}{2}}$ and the values of S_i^z limited to $-S, -S+1, \cdots, S-1, S$. We may think of our lattice as composed of N such spins, each spin precessing about a small applied field or some effective anisotropy field determining the axis of precession. The ground-state function (2) is equivalent to all spins precessing in phase, each with $S_i^z = S$. In the function (3) one of the spins has $S_j^z = S-1$. What is the equivalent classical picture of the functions (4)? Heller and Kramers[3] showed that here the spins divide up the unit loss of z component, each S_i^z being equal to $S-(1/N)$. The reason this classical picture works is that the quantum-mechanical expectation value of S_i^z in the states (4) is just $S-(1/N)$.

For large quantum number S we may take the classical picture quite literally. The many different functions (4) are then seen to correspond to the possible different phase relationships among the precessing spins. If in a given direction in the lattice each successive spin differs in phase from the preceding one by a certain amount, θ, then as the spins precess a continuous wave will be traveling through the lattice. The wave amplitude might be measured, for example, by the value of $S_i^x(t)$. This is a slightly different version of a spin wave than is Bloch's propagated spin-reversal, but it gives nearly equivalent results.

It is seen that as the phase angle θ gets larger (increasing k), more energy is required to estab-

[4] This demonstration, which was basic to Bloch's theory, was first given by J. C. Slater, Phys. Rev. 35, 509 (1930).

[5] C. Møller, Z. Physik 82, 559 (1933).

lish the spin wave; this is because neighbor spins are more out of alignment and, hence, more work must be done against exchange forces. Thus, the dispersion law (7) arises. A more detailed picture of a spin wave in a ferromagnet and a semiclassical derivation of Eq. (7) will be given in Sec. II. We shall see that the semiclassical approach gives the same results as the quantum-mechanical in the usual limit $S(S+1)\approx S^2$.

The Principle of Superposition

As the temperature is increased, more and more spin waves are excited. The basis of Bloch's approximation is that the higher eigenstates of Eq. (1′)—that is, states in which several reversals of spins are present—are very nearly combinations of functions (4), at least as long as the number of reversed spins is small compared to N. This is the principle of superposition, fundamental to all wave theories. It breaks down at a temperature of the order of one-tenth the Curie temperature. Beyond this temperature the spin-wave picture is of little merit.

The superposition principle allows any number of spin waves of a given k to be excited. This means that spin waves, in common with photons and phonons, obey Bose-Einstein statistics. Thus, the number of spin waves excited at a temperature T will be

$$n=\sum_k n_k=\sum_k\left[\exp(E_k/k_BT)-1\right]^{-1}, \quad (8)$$

where k_B is the Boltzmann constant. To evaluate n we insert the dispersion law (7) into Eq. (8) and perform the summation over the possible values of k allowed by the periodic boundary conditions. In this way it is found that n, and hence the negative change of total S^z, are proportional to $T^{\frac{3}{2}}$. Or, if M is the magnetization,

$$\Delta M(T)=M(0°\mathrm{K})-M(T)=(\mathrm{const})T^{\frac{3}{2}}. \quad (9)$$

This is Bloch's $T^{\frac{3}{2}}$ law for the low temperature dependence of magnetization of a ferromagnet. Experimental measurements by Fallot[6] verify this law and may be said to confirm the spin-wave approximation. Further details, including the value of the constant in Eq. (9) for a number of cases, may be found in Herring and Kittel.[3]

[6] M. Fallot, Ann. phys. 6, 305 (1936).

B. The Case of Antiferromagnetism

The exchange integral J is positive for ferromagnetism, negative for antiferromagnetism. This means that nearest neighbor spins in an antiferromagnet will tend to align themselves antiparallel to one another. The simplest version of this is the two-sublattice picture, in which at 0°K all the spins of one sublattice point up, all those of the other point down. For example, in a body-centered cubic all the corner sites might have spins up, all the body-center sites have spins down. This is not quite correct since the function

$$\psi_0=\alpha_1\beta_2\alpha_3\beta_4\alpha_5\cdots\beta_N, \quad (10)$$

is not an eigenfunction of the Hamiltonian (1′). However, Anderson[7] has proved that the exact ground-state energy eigenvalue must be close to the energy of the approximation (10). So we shall assume the two-sublattice picture to be approximately correct.

One can now imagine, on the semiclassical approach, a spin wave being excited in an antiferromagnet. It would appear at first glance that such a wave would be very similar to its ferromagnetic analog. Neighbor spins would be pointing antiparallel, to be sure, and the total magnetization would always be zero in the absence of an applied field; but the precessions and phase relationships should be as in ferromagnetism. Thus, one might expect the dispersion law given by Eq. (7) also to hold for antiferromagnetism. That this is not the case was first shown by Hulthén,[8] who found

$$E_k=\hbar\omega_k\cong(\mathrm{const})\ k \quad (\text{antiferromagnet}). \quad (11)$$

Recent work by Anderson[9] extending the spin-wave theory of antiferromagnetism to include the zero-point spin-wave energy has incidentally confirmed Hulthén's dispersion law and has renewed interest in the difference between ferromagnetic and antiferromagnetic spin waves. The question arises: Why should it take so much more energy (for small k) to excite an antiferromagnetic spin wave than to excite a ferromagnetic one? Since none of the mathematical

[7] P. W. Anderson, Phys. Rev. 83, 1260 (1951).
[8] L. Hulthén, Proc. Roy. Acad. Sci. (Amsterdam) 39, 190 (1936).
[9] P. W. Anderson, Phys. Rev. 86, 694 (1952); see also R. Kubo, Phys. Rev. 87, 568 (1952).

treatments of the problem seem to answer this question in a clear-cut physical way, we believe that the simple pictorial models given in this paper will be of considerable aid. From these models we shall derive the two dispersion laws (7) and (11) in a very elementary fashion.

Further, these models clarify the theory of antiferromagnetic resonance,[10] which becomes, in a sense, a branch of spin-wave theory—at least in the low temperature region where spin-wave theory is applicable.

II. SPIN WAVES IN A FERROMAGNET

The dispersion law $\omega \propto k^2$ holds for one-, two-, and three-dimensional ferromagnets, only the proportionality factor changing. For simplicity we shall consider a linear ferromagnetic chain.

Such a chain of N spins is shown in Fig. 1(a). We may imagine a small field H_0 applied in the $-z$ direction. Since the spin of an electron is directed opposite to its magnetic moment $\boldsymbol{\mu}$, the spins will all point up, as shown. We have

$$\mathbf{S}_i = \boldsymbol{\mu}_i / g\beta = \boldsymbol{\mu}_i / \hbar\gamma, \tag{12}$$

where $\beta = e\hbar/2mc$ is the Bohr magneton; $\gamma = ge/2mc$ is the magnetomechanical ratio; g is the spectroscopic splitting factor, approximately equal to 2 in ferromagnets. Owing to the negative sign of the electronic charge e, $\mathbf{S}_i$ is oppositely directed to $\boldsymbol{\mu}_i$.

At 0°K all the spins will precess in phase about z at the Larmor frequency, $\omega = \gamma H_0$. The z component of the total spin, S^z, will be a maximum. That the individual spins cannot, however, actually point in the z direction, but must have x and y components, is a well-known consequence of quantum mechanics; this has been shown by Klein and Smith[11] to lead to a zero-point spin-wave energy.

If now we excite a spin wave of wave number k in our linear chain we have the situation as shown in Fig. 1(a), which might be considered a "snapshot" of a spin wave taken at a given time. The amplitude of precession, measured by R, is so increased on the excitation of a single spin wave that the z component of total spin, S^z, decreases by one unit. The spins are also no longer in phase, the phase angle between successive spins being equal to ka, where a is the distance between spins (lattice constant). Periodic boundary conditions require the $N+1$ spin to be at the same phase as the first; hence, ka must be an integral multiple of $2\pi/N$. This is the same condition as was deduced below Eq. (6).

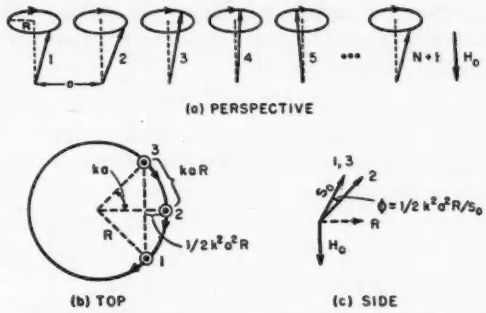


FIG. 1. Semiclassical picture of a spin wave of wave number k in a linear ferromagnetic chain of N spins. 1(a) The N precessing spins as seen in perspective at a given time. The spins are precessing clockwise as seen from the top, and each successive spin is slightly out of phase with the preceding one. 1(b) A top view of three spins collapsed onto the same precessional circle. Arrows indicate direction of precession. 1(c) A side view of three spins, also collapsed. Owing to this spin wave the precessional frequency of the spins is increased over the Larmor frequency by an amount proportional to the angle ϕ (see text). Hence, the ferromagnetic dispersion law $\omega \propto k^2$.

In Fig. 1(b) we have collapsed three spins onto the same precessional circle, as seen from the top, in order to exhibit the effect of the phase relationships. A side view, also collapsed, is given in Fig. 1(c). It is to be noted that the angle ka is very much exaggerated in the drawings; for small k we may well approximate the sagitta by $\frac{1}{2}k^2a^2R$. The angle ϕ shown in Fig. 1(c) is approximately equal to $\frac{1}{2}k^2a^2R/S_0$, since R must be small compared to the length S_0 of a spin vector in order that the semiclassical approximation obtain. Here $S_0 = (S^2+S)^{\frac{1}{2}}$, and in the semiclassical approximation $S_0 \approx S$.

We now show by classical and by quantum-mechanical arguments that, owing to the excitation of a spin wave, the increase in ω over the Larmor frequency is proportional to the angle ϕ shown in Fig. 1(c), that is, to the angle between a spin vector and the resultant vector of its two neighbor spins.

Classical Argument

In an effective magnetic field $\mathbf{H}_{eff}$ each spin, because of its associated magnetic moment, will

[10] F. Keffer and C. Kittel, Phys. Rev. **85**, 329 (1952).
[11] M. J. Klein and R. S. Smith, Phys. Rev. **80**, 1111 (1950).

experience a torque $\boldsymbol{\mu}_i \times \mathbf{H}_{\text{eff}} = \gamma\hbar \mathbf{S}_i \times \mathbf{H}_{\text{eff}}$. This torque is equal to the time rate of change of angular momentum, or

$$d(\hbar \mathbf{S}_i)/dt = \gamma\hbar \mathbf{S}_i \times \mathbf{H}_{\text{eff}}. \tag{13}$$

We may identify $\mathbf{H}_{\text{eff}}$ from the Hamiltonian of a single spin, including the Zeeman term,

$$\begin{aligned}\mathcal{H}_i &= -g\beta \mathbf{S}_i \cdot [\mathbf{H}_0 + (2J/g\beta)\textstyle\sum_j \mathbf{S}_j] \\ &= -g\beta \mathbf{S}_i \cdot \mathbf{H}_{\text{eff}}.\end{aligned} \tag{14}$$

The j summation is over the nearest neighbors to i.[12] From Eqs. (13) and (14) we have

$$d(\hbar \mathbf{S}_i)/dt = \gamma\hbar \mathbf{S}_i \times [\mathbf{H}_0 + (2J/g\beta) \textstyle\sum_j \mathbf{S}_j]. \tag{15}$$

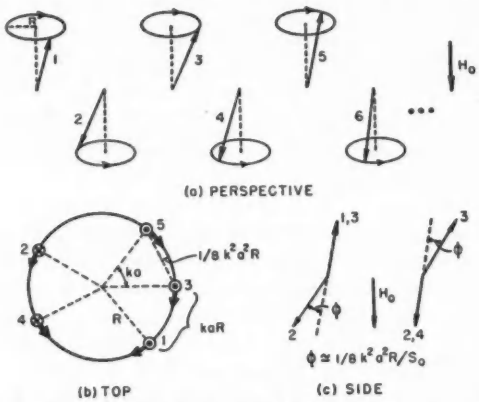


FIG. 2. Semiclassical picture of spin waves in an antiferromagnet. 2(a) The two sets of spin waves 1, 3, 5··· and 2, 4, 6··· as seen in perspective at a given time. 2(b) Top view collapsed showing phase relationships. The arrows indicate precessional directions in absence of Larmor precession. 2(c) Two side views, the first in the plane made by 2 and the resultant of 1 and 3, the second in the plane made by 3 and the resultant of 2 and 4. Since the exchange forces make the down spins precess oppositely to the up spins (see text), this is not the correct picture for an antiferromagnetic spin wave.

Quantum-Mechanical Argument

We follow Herring and Kittel.[3] The quantum equation of motion for $\mathbf{S}_i$ is

$$\begin{aligned}i\hbar(d\mathbf{S}_i/dt) &= [\mathbf{S}_i, \mathcal{H}] \\ &= [\mathbf{S}_i, -g\beta \textstyle\sum_j \mathbf{S}_j \cdot \mathbf{H}_0 - 2J \sum_{(j,k)} \mathbf{S}_j \cdot \mathbf{S}_k] \\ &= [g\beta(\textstyle\sum \mathbf{S}_j \cdot \mathbf{H}_0) + 2J(\sum \mathbf{S}_j \cdot \mathbf{S}_k)]\mathbf{S}_i \\ &\qquad - \mathbf{S}_i[g\beta(\textstyle\sum \mathbf{S}_j \cdot \mathbf{H}_0) + 2J(\sum \mathbf{S}_j \cdot \mathbf{S}_k)] \\ &= \textstyle\sum_j \{[g\beta(\mathbf{S}_i \cdot \mathbf{H}_0) + 2J(\mathbf{S}_i \cdot \mathbf{S}_j)]\mathbf{S}_i \\ &\qquad - \mathbf{S}_i[g\beta(\mathbf{S}_i \cdot \mathbf{H}_0) + 2J(\mathbf{S}_i \cdot \mathbf{S}_j)]\} \\ &= [\mathbf{S}_i \times \mathbf{S}_i] \times [g\beta \mathbf{H}_0 + 2J \textstyle\sum_j \mathbf{S}_j].\end{aligned}$$

[12] In obtaining the total exchange Hamiltonian of Eq. (1) from summing Eq. (14) over i, one must be careful to sum only over combinations of nearest neighbors in order not to count mutual energy terms twice.

On using the commutation relation $\mathbf{S} \times \mathbf{S} = i\mathbf{S}$ and the relation $g\beta = \hbar\gamma$ we obtain the classical result, Eq. (15). The classical and quantum-mechanical equations of motion are the same due to the well-known relationship between commutators and Poisson brackets.

The right-hand side of Eq. (15) is the torque acting on a spin i. If the spins precess together about the z axis, always maintaining the same phase relationships, this torque remains constant in magnitude throughout the precession. Such a mode of motion must exist due to symmetry of the Hamiltonian about z; we call this motion a *normal mode*. From Eq. (15) and Fig. 1(c) the magnitude of the torque is easily found to be

$$\begin{aligned}T &= \hbar\gamma S_0[H_0 \sin(\mathbf{S}_i, \mathbf{H}_0) - (2J/g\beta)2S_0 \sin\phi] \\ &= \hbar\gamma[RH_0 - (2J/g\beta)S_0 a^2 k^2 R].\end{aligned} \tag{16}$$

It should be noted that both S_0 and R as used here are dimensionless. From Eq. (15) one finds the angular velocity of precession due to this torque,

$$\hbar\omega R = T; \tag{17}$$

and hence,

$$\omega \cong \gamma H_0 - (2J/\hbar)S_0 a^2 k^2. \tag{18}$$

We have used the right-hand-screw sign conventions: T positive when into the plane of Fig. 1(c); likewise ω positive when clockwise about $+z$. As γ is negative the precession will be clockwise about $-z$, or when seen from the top as in Fig. 1(b).

Since the excitation of a spin wave reduces S^z, and since H_0 is along $-z$, it is seen that the energy in the spin wave (18) is $-\hbar\omega$.

We have thus given a simple derivation of the dispersion law, Eq. (7), to which we have added a Larmor term. We note that the factor S in the constant of Eq. (7) is replaced by S_0 in Eq. (18), as was to have been expected from a semiclassical approximation.

Ferromagnetic Resonance

In case a spin wave with $k=0$ is excited, the amplitude of precession will change but the phase relationships will remain unchanged. From Eq. (18) it is seen that the frequency of such a spin wave is the Larmor frequency. Excitation of $k=0$ spin waves takes place in ferromagnetic resonance, in which an rf field is applied per-

pendicular to H_0 and of such wavelength that all spins remain in phase. Our pictorial representation makes it clear why the exchange interaction has no effect on the ferromagnetic resonance frequency, which in the absence of demagnetizing fields is simply the Larmor frequency.

III. SPIN WAVES IN AN ANTIFERROMAGNET

Spin waves in an antiferromagnet are shown in Fig. 2(a). We may begin by setting up two sets of spin waves, one in the sublattice pointing up (spins 1, 3, 5, etc.), and one in the sublattice pointing down (spins 2, 4, 6, etc.). These sets, however, are not independent, owing to the exchange interaction between nearest neighbor spins.

In Fig. 2(b) a top view is given of several spins collapsed onto the same precessional circle. Side views, also collapsed, are given in Fig. 2(c).

We apply Eq. (15) to an up spin and a down spin, obtaining with the aid of Fig. 2(c) two equations analogous to Eq. (16):

$$T_u = \hbar\gamma[RH_0 + (J/2g\beta)S_0a^2k^2R],$$
$$T_d = \hbar\gamma[-RH_0 + (J/2g\beta)S_0a^2k^2R]. \quad (19)$$

Here T_u and T_d are the magnitudes of the torques on an up spin and on a down spin, respectively. Again the sign convention is T positive when into the plane of Fig. 2(c). Maintaining the convention ω positive when clockwise about $+z$, we have

$$\hbar\omega_u R = T_u,$$
$$-\hbar\omega_d R = T_d. \quad (20)$$

Thus, the angular velocities of precession are

$$\omega_u = \gamma H_0 + (J/2\hbar)S_0a^2k^2,$$
$$\omega_d = \gamma H_0 - (J/2\hbar)S_0a^2k^2. \quad (21)$$

But the exchange parts of these precessional frequencies are in opposite directions! The spins will not precess together about the z axis always maintaining the same phase relationships, as was postulated in assuming the magnitude of torque (19) was a constant of the motion. However, as in the ferromagnetic case, the Hamiltonian has symmetry about z; and hence, normal modes must exist in which phase relationships are maintained.

The Normal Modes

It is easiest to visualize the normal modes in the absence of H_0. Then, as seen from the top, *all* spins will precess clockwise (mode 1), or *all* spins will precess counterclockwise (mode 2). As we shall see, these modes can be achieved only by having the up spins precess in a larger (mode 1) or in a smaller (mode 2) circle than do the down spins. The normal mode 1 is illustrated in Fig. 3. The normal mode 2 would be illustrated by Fig. 3(a) turned upside down, except that the pres-

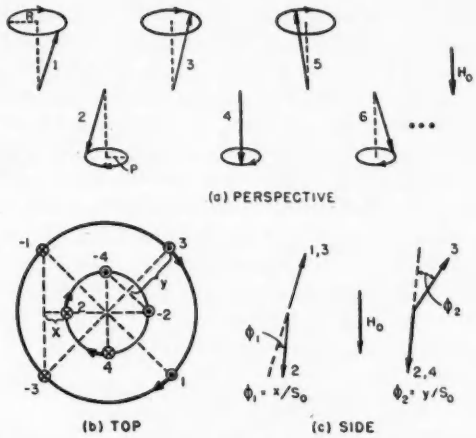


FIG. 3. Semiclassical picture of normal mode 1 of spin waves in an antiferromagnet. The up spins precess in circles of radius R, the down spins in circles of radius P, with $R>P$. 3(a) As seen in perspective at a given time. 3(b) Top view collapsed showing phase relationships. Positions marked -1, -2, -3, -4 are those of imaginary vectors directed opposite to spins 1, 2, 3, 4, respectively. Arrows indicate precessional directions, which are now all clockwise. 3(c) Two side views. Note that 2 is on the opposite side of the line made by 1 and 3 as compared with Fig. 2(c). This is why the down spins, 2 and 4, are now precessing clockwise. This mode leads to antiferromagnetic dispersion law $\omega \propto k$.

ence of H_0 causes the two modes to be slightly different. That is, the Larmor precession is always clockwise about H_0, seen from the top, and hence will add to the clockwise precession of mode 1 but subtract from the counterclockwise precession of mode 2.

In the absence of H_0 the two normal modes are degenerate in energy and various linear combinations could be considered as "normal" modes. One such combination was obtained by Anderson[9] when he used a canonical transformation to put his Hamiltonian into normal coordinate form

for two sets of harmonic oscillators. The canonical transformation is a mathematical way of making the transition from Fig. 2 to Fig. 3. That Anderson's transformation is not the correct one in the presence of H_0 has been demonstrated by one of us, and the correct transformation has been given.[13]

In Fig. 3(b) we show a top view of the precessional circles. Shown on the circles are the phase positions of the spins 1, 2, 3, 4 and of the corresponding oppositely directed -1, -2, -3, -4. Side views are given in Fig. 3(c). It is clear from the figures that 1 and 3 act on 2 through an angle ϕ_1 determined by x, whereas 2 and 4 act on 3 through a slightly larger angle ϕ_2 determined by y. It is also to be noted that the vector cross product of the resultant of 1 and 3 with 2 is oppositely directed to its analog in Fig. 2(c). This accounts for the fact that the down spins now precess in the same direction as the up spins, and hence, we have a normal mode.

We have a normal mode, that is, if the angular velocities of precession of the down spins is the same as that of the up spins. We now find these velocities. The torque equation (15) yields

$$\begin{aligned} T_u &= \hbar\gamma[RH_0+(2J/g\beta)2yS_0], \\ T_d &= \hbar\gamma[-PH_0+(2J/g\beta)2xS_0], \end{aligned} \tag{22}$$

where R and P are the radii of the circles of precession of the up and down spins, respectively. Instead of Eq. (20) we now have

$$\begin{aligned} \hbar\omega_u R &= T_u, \\ -\hbar\omega_d P &= T_d. \end{aligned} \tag{23}$$

On equating ω_u and ω_d as obtained from inserting Eq. (23) into Eq. (22), we have as the condition for a normal mode

$$y/R = x/P. \tag{24}$$

Since the phase angle between spins 1 and 3 is ka, we have from Fig. 3(b)

$$x = R - P - (k^2a^2R/8); \tag{25a}$$

similarly,

$$y = R - P + (k^2a^2P/8). \tag{25b}$$

From Eqs. (24) and (25)

$$R - P = ka[(R^2+P^2)/8]^{\frac{1}{2}}, \tag{26}$$

and after a bit of algebra

$$\frac{y}{R} = \frac{x}{P} = \frac{ka(P+R)}{(8R^2+8P^2)^{\frac{1}{2}}}.$$

In this expression we may set $R=P$, for small k, by virtue of Eq. (26); thus,

$$y/R = x/P \cong ka/2. \tag{27}$$

Finally, from Eqs. (22), (23), and (27)

$$\omega_u^{(1)} = \omega_d^{(1)} = \gamma H_0 + (2J/\hbar)S_0ak \quad \text{(mode 1)}. \tag{28a}$$

Similarly, we obtain

$$\omega_u^{(2)} = \omega_d^{(2)} = \gamma H_0 - (2J/\hbar)S_0ak \quad \text{(mode 2)}. \tag{28b}$$

In determining the direction of these precessions one must recall that both γ and J are negative. Sign conventions used here are given below Eq. (18).

Thus, we have demonstrated the dispersion law for antiferromagnets, Eq. (11).

It should be noted that the energy of a spin wave in mode 1 is $-\hbar\omega^{(1)}$ whereas in mode 2 it is $+\hbar\omega^{(2)}$. This will be clarified in Sec. IV.

IV. ANTIFERROMAGNETIC RESONANCE

On exciting a spin wave of mode 1 we reduce the z component of the total spin of the up sublattice by more than we increase the z component of total spin of the down sublattice. It can be shown[13] that the net effect is to *decrease* the z component of total spin of the entire lattice, S^z, by $\hbar$. Similarly the net effect of exciting a spin wave of mode 2 is to *increase* S^z by $\hbar$. Thus,

$$S^z = \sum_k (n_{2k} - n_{1k}), \tag{29}$$

where n_{2k} and n_{1k} are the number of spin waves of wave number k excited in mode 2 and mode 1,

[13] F. Keffer, thesis, Berkeley, January 1952. We repeat here the transformation. Anderson's Eq. (15) (see reference 9) is replaced by

$$\begin{aligned} P_\lambda &= -\eta(1-\eta)^{-1}p_{1\lambda}+(1-\eta)^{-1}p_{2\lambda}, \\ Q_\lambda &= \eta(1+\eta)^{-1}q_{1\lambda}+(1+\eta)^{-1}q_{2\lambda}, \\ R_\lambda &= (1+\eta)^{-1}q_{1\lambda}+\eta(1+\eta)^{-1}q_{2\lambda}, \\ S_\lambda &= (1-\eta)^{-1}p_{1\lambda}-\eta(1-\eta)^{-1}p_{2\lambda}, \end{aligned}$$

where

$$\eta = -\sigma_\lambda - |(\sigma_\lambda^2-1)^{\frac{1}{2}}|,$$

with, using Anderson's notation,

$$\sigma_\lambda = [1+(K/DJ)]/\gamma_\lambda.$$

The normal modes in the presence of H_0 have also been derived (using a different formalism) by J. M. Ziman, Proc. Phys. Soc. (London) A65, 540 (1952); and by T. Nakamura, Prog. Theoret. Phys. 7, 539 (1952).

respectively. This equation may be obtained by an extension of Anderson's work[9] on use of the correct transformation in the presence of H_0.[13,14]

The statement about the energies of the spin waves made at the end of Sec. III becomes clear in the light of Eq. (29). Since H_0 is directed along $-z$, energy is required to increase S^z.

As in ferromagnetic resonance, discussed in Sec. II, an rf field applied perpendicular to H_0 will excite only $k=0$ spin waves. It will excite equal numbers in mode 1 and mode 2; hence, S^z cannot change, and no energy will be required. That is, an indefinite number of spin waves of $k=0$ will be excited in the presence of no rf energy whatever! The spins will constantly be turning over. This has been pointed out by Anderson,[9] who introduces some anisotropy to keep the sublattices from interchanging.

Probably anisotropy is always present in antiferromagnetism. In MnF_2, for example, one of us[15] has calculated an effective anisotropy field of 8800 oersteds tending to keep the spins lined up. We need to modify our spin-wave picture to take account of this. We introduce an effective anisotropy field in the negative z direction acting on the up spins (whose magnetic moments are down), and one in the positive z direction acting on the down spins. Just as with the exchange fields of Fig. 2 these fields will tend to make the members of the two sublattices precess in *opposite* directions. In order that the spins precess in the same direction they must precess in different size circles. Thus, whether or not the spins are all in phase, i.e., even for $k=0$ spin waves, the spins will be inclined to one another. To perform this inclination requires work against the exchange field H_E as well as against the anisotropy field H_A. We use the notation $H_E=|(2J/g\beta)\sum_j \mathbf{S}_j|$; see Eq. (14). It has been shown by one of us using Anderson's formalism[13] and by one of us using the semiclassical formalism[16] that this amount of work (for small k) is equal to [17]

$$E_k=\mp\hbar\omega_k=\mp\hbar\gamma H_0-\hbar\gamma(2H_EH_A+H_A^2+\text{const}\ a^2k^2)^{\frac{1}{2}}. \quad (30)$$

In $\mp$ the $-$ is for mode 1, the $+$ for mode 2. The value of the constant for a three-dimensional simple cubic is $H_E^2/12$; for a one-dimensional chain it is $H_E^2/4$. For $H_A=0$, Eq. (30) reduces to Eq. (28).

To excite a spin wave of $k=0$ in a resonance experiment we must match the precessional frequency given by Eq. (30),

$$\omega_0=\gamma H_0\pm\gamma(2H_EH_A+H_A^2)^{\frac{1}{2}}. \quad (31)$$

This resonance frequency has been derived from a molecular field approach by Kittel[18] and by Nagamiya.[19] The normal modes for resonance, very similar to the normal modes shown here in Fig. 3, are given in Fig. 1 of Keffer and Kittel.[10]

ACKNOWLEDGMENTS

We are indebted to Professor Kittel for many stimulating discussions of the theory of spin waves. We wish to thank Dr. P. W. Anderson for the privilege of seeing his manuscript prior to publication.

[14] It is to be noted that Eq. (29) cannot be obtained from Anderson's transformation.

[15] F. Keffer, Phys. Rev. **87**, 608 (1952).

[16] H. Kaplan, Phys. Rev. **86**, 121 (1952), and thesis, Berkeley, August 1952.

[17] See the remarks at the end of Sec. III relating frequencies to energies in the two modes.

[18] C. Kittel, Phys. Rev. **82**, 565 (1951).

[19] T. Nagamiya, Prog. Theoret. Phys. **6**, 342 (1951).

Indiana Section

The Spring Meeting of the Indiana Section of the American Association of Physics Teachers will be held at Ball State Teachers' College, Muncie, Indiana, on Saturday, May 23, 1953.

Čerenkov Radiation in a Dispersive Medium

H. Motz and L. I. Schiff
*Microwave Laboratory and Department of Physics, Stanford University, Stanford, California**
(Received October 10, 1952)

Attention is called to the disagreement between a recent statement of Sommerfeld that the direction of propagation of Čerenkov radiation in a dispersive medium is determined by the group velocity, and the generally accepted result that the phase velocity must be used. The latter is shown to be correct, and the role of the group velocity is discussed in a way similar to that given in Tamm's 1939 paper on the theory of the Čerenkov radiation.

In a recently published book on optics, Sommerfeld[1] states that the direction of propagation of Čerenkov radiation in a dispersive medium is given by the usual formula[2] if the velocity of light in the medium is taken to be the group, not the phase velocity. This is contrary to the generally accepted result, according to which the phase velocity must be used. The difference between these two results can be of considerable physical significance in an experiment like that recently performed by Mather.[3] That the use of the phase velocity is actually correct can readily be seen in the following way. For simplicity, we restrict our attention to an infinitesimal range of radiated frequencies; this might be achieved in practice by use of a narrow-band filter. Then the usual calculation shows that the Poynting vector for each monochromatic component has the direction calculated from the phase velocity. All of these Poynting vectors are infinitesimally close together, so that the direction of propagation of the entire range of frequencies is infinitesimally close to that calculated for any one frequency component in the range by use of the phase velocity.

It is still of some interest, however, to see as clearly as possible the role played by the group velocity. This point has already been discussed by Tamm[4] in his classical paper on the theory of the Čerenkov radiation. He shows that for a narrow range of frequencies, the radiation at any instant is concentrated in a thin conical shell whose vertex is at the moving charge, and whose half-angle of opening θ is given by

$$\cot\theta = [(n_0 v/c)^2 - 1]^{\frac{1}{2}} + \omega_0 n_0 (v/c)^2 (dn/d\omega)_0 [(n_0 v/c)^2 - 1]^{-\frac{1}{2}}, \quad (1)$$

where v is the velocity of the charge, c the velocity of light in vacuum, $n(\omega)$ the refractive index of the medium at angular frequency ω, and the frequency range is centered at ω_0 with $n_0 \equiv n(\omega_0)$. If the radiation were to propagate in directions perpendicular to the planes tangent to the surface of this cone, it would agree with the correct theory only for a nondispersive medium ($dn/d\omega = 0$). Actually, the radiation propagates in directions that make the angle θ_0 with the path of the charge, where

$$\tan\theta_0 = [(n_0 v/c)^2 - 1]^{\frac{1}{2}}, \quad (2)$$

so that the cone may be thought of as "side-slipping" as it moves along with the charge.

It seems worth while to recast Tamm's argument in a slightly different form, which brings out the analogy with the well-known derivation of the group velocity of a one-dimensional wave packet.[5] For simplicity, we regard the field as varying only in a plane through the path of the charge, so that only two dimensions need be used. A general wave packet solution for any field component is then

$$u(x, y, t) = \int\int A(k_x, k_y) \times \exp[i(k_x x + k_y y - \omega t)] dk_x dk_y, \quad (3)$$

where $\omega = c(k_x^2 + k_y^2)^{\frac{1}{2}}/n(\omega)$; as before, the inte-

* The work reported herein was performed in part under a contract between Stanford University and the U. S. Office of Naval Research.

[1] A. Sommerfeld, *Vorlesungen über Theoretische Physik*, Band IV, Optik (Dieterich Verlag., Wiesbaden, 1950), p. 341.

[2] See, for example, reference 4, or L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 264.

[3] R. L. Mather, Phys. Rev. **84**, 181 (1951).

[4] I. Tamm, J. Phys. U.S.S.R. **1**, 439 (1939); see especially Section 5.

[5] See, for example, M. Born, *Atomic Physics* (Blackie and Son, Ltd., Glasgow, 1946), p. 295.

gration is over a narrow range of frequency or propagation vector. We expand the integrand about the center of the range ω_0, $\mathbf{k}_0$, and rewrite Eq. (3) as the product of a slowly varying amplitude and a rapidly varying phase factor:

$$u(x, y, t) = B(x, y, t) \times \exp[i(k_{0x}x + k_{0y}y - \omega_0 t)],$$

$$B(x, y, t) = \int\int a(k_{1x}, k_{1y}) \times \exp[i(k_{1x}x + k_{1y}y - \omega_1 t)] dk_{1x} dk_{1y}, \tag{4}$$

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{k}_1, \quad \omega = \omega_0 + \omega_1,$$

$$a(k_{1x}, k_{1y}) = A(k_x, k_y).$$

The amplitude B is a slowly varying function of x, y and t since $\mathbf{k}_1$ and ω_1 are restricted to narrow ranges of values. The energy is concentrated in regions where B is large; the motion of these regions may be found by writing down the condition that B = constant, which has the differential form

$$B_x dx + B_y dy + B_t dt = 0, \tag{5}$$

where $B_x \equiv \partial B/\partial x$, etc. Equation (5) describes a series of parallel lines of slope $-B_x/B_y$; the intercept of any one of these with the x axis moves with the velocity $-B_t/B_x$, and the y axis intercept with velocity $-B_t/B_y$. From Eq. (4), we have

$$B_x = i\int\int k_{1x} a \exp(\quad) dk_{1x} dk_{1y},$$

$$B_y = i\int\int k_{1y} a \exp(\quad) dk_{1x} dk_{1y}, \tag{6}$$

$$B_t = -i\int\int \omega_1 a \exp(\quad) dk_{1x} dk_{1y}.$$

So far, the wave packet treatment is quite general. In the Čerenkov case, $A(k_x, k_y)$ vanishes unless $k_y = k_x[(nv/c)^2 - 1]^{\frac{1}{2}}$, so that $\omega = k_x v$. This follows from the usual theory, or may be seen physically from the requirement that the radiation pattern be stationary with respect to the charge moving in the x direction, which means

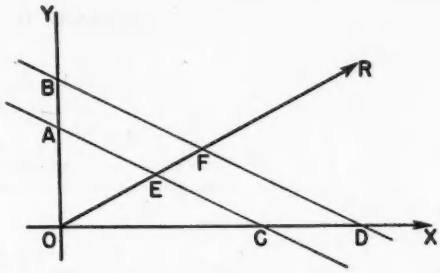


FIG. 1. A charge moves in the direction OX, and a line of constant radiation amplitude moves from AC to BD in time Δt. The direction of energy flow OR is such that the distance EF is equal to Δt times the group velocity of the radiation.

that the phase velocity component in that direction, ω/k_x, must be equal to v. Then in the first of Eqs. (6), k_{1x} can be replaced by ω_1/v, and in the second k_{1y} can be replaced approximately by

$$\omega_1[d\{(\omega/v)[(nv/c)^2 - 1]^{\frac{1}{2}}\}/d\omega]_0 = (\omega_1/v)\{[(n_0 v/c)^2 - 1]^{\frac{1}{2}} + \omega_0 n_0 (v/c)^2 (dn/d\omega)_0 [(n_0 v/c)^2 - 1]^{-\frac{1}{2}}\}.$$

These substitutions lead to

$$B_x = -B_t/v,$$
$$B_y = -(B_t/v)\{[(n_0 v/c)^2 - 1]^{\frac{1}{2}} + \omega_0 n_0 (v/c)^2 (dn/d\omega)_0 [(n_0 v/c)^2 - 1]^{-\frac{1}{2}}\}.$$

Thus the lines (actually generators of cones) of constant amplitude are inclined to the x-axis with the angle θ given by Eq. (1).

The direction of flow of the energy may be found from the construction of Fig. 1. The charge moves in the direction OX, and a line of constant amplitude has the position AC at one instant of time and the position BD at a time Δt later. From the discussion of Eq. (5), we have $AB = -(B_t/B_y)\Delta t$, and $CD = -(B_t/B_x)\Delta t$. The radiated energy must move in the direction OR, which is so determined that its point of intersection with the line of constant amplitude moves with the group velocity u of the radiation; that is, the distance EF must be equal to $u\Delta t$. Now $\omega n(\omega) = ck$, where k is the magnitude of the propagation vector, and $u = d\omega/dk$; therefore $u = c/[n_0 + \omega_0 (dn/d\omega)_0]$. It can then be shown quite readily that the angle ROX in Fig. 1 is just equal to the angle θ_0 given by Eq. (2).

Behavior of Magnetic Materials

R. M. Bozorth
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 23, 1952)

This is a review of recent work in which the atomic theory of ferromagnetism and the domain theory of magnetization are applied to new materials.

RECENTLY two new types of magnetic materials have come into prominence, which illustrate very well two important problems in ferromagnetism that have existed for some time. These problems are:

(1) What is the atomic structure of magnetic materials, and what relation does this structure have to magnetic properties?

(2) What is the mechanism of magnetization whereby the magnetization of a material is changed by application of a magnetic field?

These will be discussed in relation to two new kinds of materials: (a) ferrites, and (b) fine particle magnets, both of which are of considerable commercial importance as well as of scientific interest.

First, some general remarks on atomic structure are needed. For a material to be ferromagnetic its atoms must possess a permanent magnetic moment; that is, each atom must be a small magnet. Also, there must be some sort of interatomic force which maintains the moments of neighboring atoms parallel; otherwise, the material would be paramagnetic, that is, only weakly magnetic.

Figure 1 shows the structure of an iron atom. It is well known to be built up of electron shells, as shown, and in each shell the electrons have both orbital motions around the atomic nucleus and spins about their own electron axes. The first two shells are complete. They contain as many electrons as these shells can ever contain, and the magnetic moments of the electrons in each shell are balanced so that the net magnetic moment is zero. However, in the third shell one of the subshells is completely filled, whereas its counterpart is only partially filled, so that we have a net magnetic moment corresponding to the spin of $5-1$ or 4 electrons. This atom then has an atomic moment of 4 Bohr magnetons.

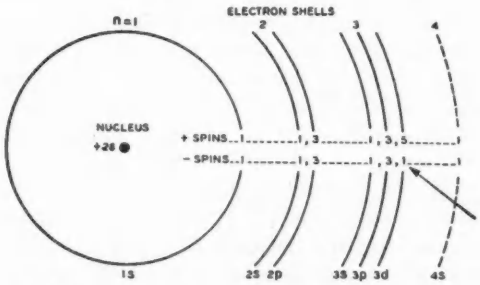


FIG. 1. Electron shells in an iron atom, showing location of unfilled $3d$ shell responsible for the magnetic moment of the atom, and for the ferromagnetism of the metal.

This is the well-established structure of a free atom of iron. When this atom becomes part of a solid metallic structure, the distribution of electrons is disturbed, and we know by experiment that each atom has on the average a magnetic moment of 2.2 Bohr magnetons. We do not understand the structure of a metal well enough to know why this number should be what it is.

Relatively little is known about the interatomic forces that maintain the moments of neighboring atoms parallel. However, Heisenberg showed in 1928 that it is a quantum-mechanical exchange force which depends on interatomic distance. In 1930 Slater found that the distance, D, between neighboring atoms, divided by the diameter, d, of the incomplete shell, in which the magnetic moment resides, is larger in the ferromagnetic elements than in the nonmagnetic ones. Bethe suggested that the atomic moments in a crystal are maintained parallel by quantum-mechanical forces only when they are separated by more than a minimum distance, and that when the distance increases beyond this the forces become first larger and then smaller again. However, the form of this interaction curve, relating the force of ferromagnetism to atomic distance, is not at all well established.

DOMAINS

Now, as neighboring atoms are aligned parallel, why is not any piece of ferromagnetic material completely and permanently magnetized in one direction? In 1907 Weiss suggested an explanation which is illustrated in Fig. 2(a) and which in principle has been very well substantiated since its proposal in 1907. According to this idea small regions or *domains* are magnetized to saturation in various directions so that the net magnetization of the whole specimen is zero, before a magnetic field has been applied.

Since Weiss' time we have come to know that there are certain directions in a crystal which are preferred, and in iron we know that these directions are the six directions which lie parallel to the cubic axes of the crystal. A revised domain picture then is shown in Fig. 2(b), where the directions of the domains in a single crystal are limited but still the net magnetic moment of the whole specimen is zero.

Recently H. J. Williams[1] and his colleagues have been able to see under the microscope the actual domains in a crystal specimen and detect the direction of magnetization in each domain. A photograph of this kind is shown in Fig. 2(c), and other specimens are shown in some of the following figures. The technique used is similar in principle to the detection of cracks or flaws in a large piece of iron by sprinkling iron filings over it. When a colloidal suspension of Fe_3O_4 is placed on a crystal whose surface is carefully prepared, the particles of Fe_3O_4 collect along domain boundaries where there are small stray fields, and they appear under the microscope as fine light lines.

When a field is applied, the domain structure of Fig. 2(c) changes to that of Fig. 2(d); that is, the zig-zag domain wall shifts in position. This is the well-known mode of magnetization in weak fields—a change in magnetization takes place by movement of the boundary separating domains so that the domains which are aligned more favorably with respect to the field (those above the line) are enlarged at the expense of other domains (those below). Such changes have been shown in a striking way in moving pictures.

[1] Williams, Bozorth, and Shockley, Phys. Rev. **75**, 155 (1949).

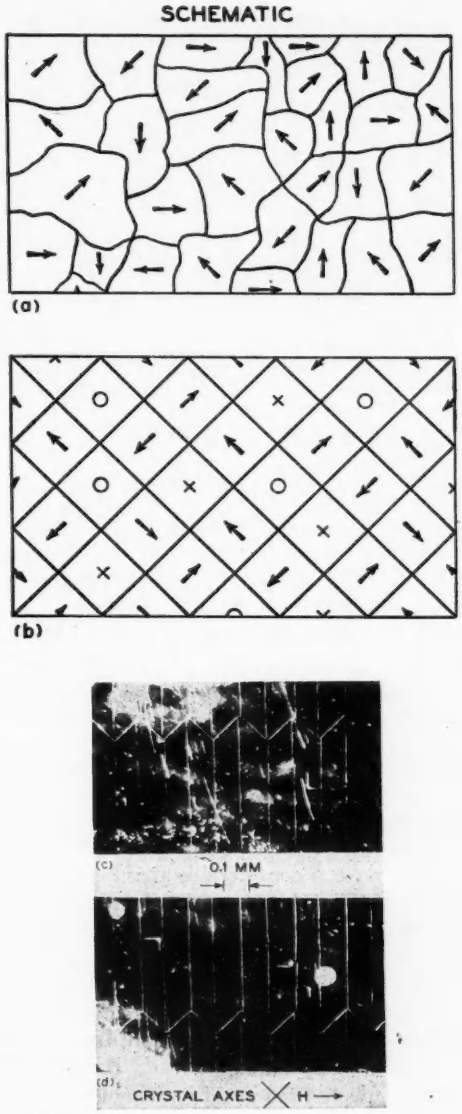


FIG. 2. Domain diagrams (a) and (b), and actual photographs (c) and (d), of domains in an iron crystal (Williams).

Many domain structures have been observed under the microscope, and we can interpret many of them, although not all. We are, however, quite confident that we know the principles which govern domain formation, and these are illus-

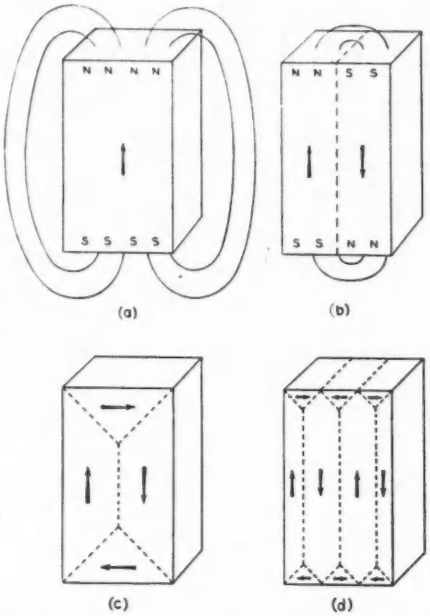


FIG. 3. Possible domain structures, showing large magnetostatic energy in (a), and successively lower energy in other structures in which there is wall energy (b) and magnetostrictive energy (c); (d) represents the kind of structure actually observed (Fig. 4).

trated in Fig. 3.[2] If we have a block of magnetic material as shown in Fig. 3(a) and if this is magnetized as a single domain, then we will have a large external field as indicated by the curved lines. Of course, a considerable amount of energy is stored in this field. However, if we divide this block into two portions by a domain wall so that the two halves are magnetized in opposite directions, the energy in the external field is obviously greatly reduced. We can also make the domain structure shown in Fig. 3(c), where the exterior field is completely eliminated and the domains are always magnetized in a direction of easy magnetization parallel to a crystal axis. But here an additional energy is present. Besides the energy associated with a domain wall, which in iron is about 1 or 2 ergs per square centimeter, we have energy associated with magnetostriction. When a domain becomes magnetized it changes its dimensions slightly by a few parts per million, and the domains which are magnetized in a horizontal direction in the figure do not quite fit into the spaces allotted to them. As a result the structure is slightly strained, and we can calculate the amount of energy associated with this strain. In the structure shown in Fig. 3(d) this magnetostriction energy has been reduced considerably by reducing the amount of strained material, but the additional wall area has increased the energy associated with it.

The stable structure which is produced in nature is, of course, that for which the total energy is minimum. This requires a calculable balance between the volumes with horizontal magnetization and those with vertical magnetization. The scale of the structure derived by theory is actually very close to that observed. Figure 4 shows the powder pattern of the domain structure observed in an actual crystal, and its resemblance to Fig. 3(d) is obvious.

Another type of domain pattern is shown in cobalt. This material has a hexagonal crystal structure, and magnetic measurements show that the hexagonal axis is a direction of easy magnetization. Unlike cubic iron, which has six directions of stable magnetization, cobalt has just two (antiparallel) directions. Figure 5(a) shows a surface of a cobalt crystal which has been cut so that it contains the crystal axis. The domain structure of this material is like a bundle of needles with the axes of the needles all parallel

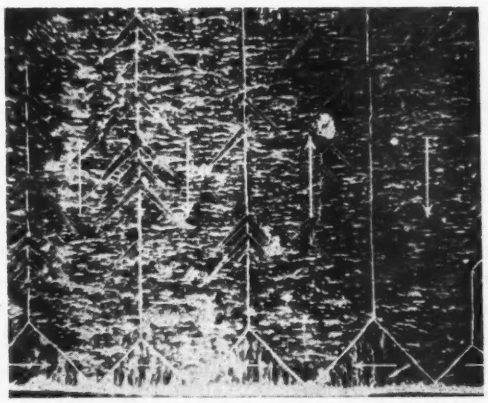


FIG. 4. Observed magnetic domain pattern corresponding to Fig. 3(d). Edge of crystal shows near bottom of figure. Length of specimen shown is about 0.6 mm. Retouched; arrows added to show local direction of magnetization.

[2] R. M. Bozorth, *Ferromagnetism* (D. Van Nostrand Company, Inc., New York, 1951). This book may also be consulted for further information about various topics discussed in this article.

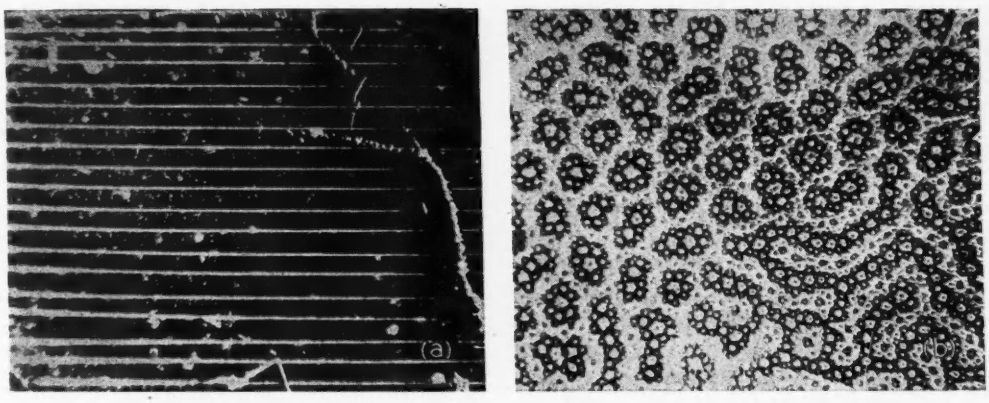


FIG. 5. Domain patterns of cobalt on surfaces parallel (a) and perpendicular (b) to hexagonal axis, showing rod-like domain structure.

and the magnetization of neighboring domains lying in the two antiparallel directions. A domain pattern, Fig. 5(b), on a surface cut so that it is normal to the axis of the crystal, shows the ends of the needle-like domains, as one would expect.

FINE PARTICLES

An interesting application of domain theory is concerned with the size of the individual specimen. If we take a cube of the material, we may assume either of two configurations as shown in Fig. 6. The first diagram shows the material as a single domain magnetized vertically. The second shows the cube composed of several domains with walls separating them. In the first case we have a large amount of magnetostatic energy associated with the exterior field arising from the north and south poles on the surfaces. The second diagram has no exterior field but has an energy associated with the domain walls. One can readily calculate the relative energies of the two configurations, and it is found that the magnetostatic energy E_1 of the first cube is

$$E_1 \approx 2I_s^2 d^3,$$

and the wall energy E_2 of the second is

$$E_2 \approx 2\sqrt{2}\gamma d^2,$$

I_s being the intensity of magnetization at saturation, d the length of the edge of the cube and γ the energy of 1 cm² of wall. The first is proportional to the third power of the cube edge, the second to the square of the edge. Normally in large particles the second configuration is more stable, and several domain walls are present. However, if the particle size is small enough, E_1 will be less than E_2 and the first configuration will be more stable, and so each particle will be a single domain. In iron the two energies are equal when the particles are about 300A in size.

Now consider what happens to these different domain structures when a field is applied. When domain boundaries are present, application of a field will move the boundaries as shown in the upper half of Fig. 7, and this process will proceed in relatively low fields with great ease. However, if we apply a field to a single domain particle, this process cannot occur because no boundaries exist. Therefore, we must apply a field strong enough to rotate the magnetization out of the direction of easy magnetization. The field required for this is determined by the nature of the crystal. In a cubic crystal like iron the energy required to rotate the magnetization is shown

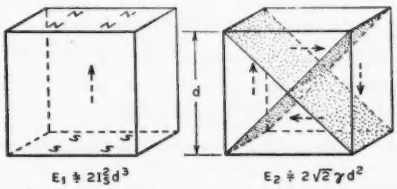


FIG. 6. Two possible domain structures for a single particle of iron: (a) stable in small particles, (b) in particles larger than about 300A, when $E_1 \approx E_2$.

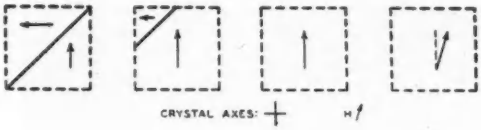


FIG. 7. Top portion shows mechanism of change of magnetization by displacement of domain wall and then by rotation of magnetization. Lower diagram shows mechanism when walls do not exist and magnetization must be rotated over energy hump caused by crystal anisotropy.

qualitatively by the clover-leaf pattern at the lower left of Fig. 7, which is consistent with the cubic symmetry of the system. The scale of the pattern is measured by the magnetic anisotropy constant, K, which can readily be determined experimentally. In iron the anisotropy is such that about 500 oersteds are required to rotate the magnetization from an easy direction to a hard direction. In a specimen containing many domains, on the contrary, a change in magnetization can be produced by a shift in the domain walls, and these move readily in a field of about 1 oersted.

The practical result of all this is that if we form iron into very fine particles, of the order of 200 to 300 angstroms in diameter, we have made tiny permanent magnets of it. Such magnets have been made on an experimental basis by careful reduction of the metal oxide, and these have a coercive force of about 500 oersteds. Also, magnets made of pressed iron powder have been made on a commercial scale with quality that compares favorably with the best permanent magnets that were known 20 years ago. So far they cannot compete with our best modern alloys, but other materials in powder form have the possibility of exceeding our best commercial materials in quality. This field is developing rapidly, and it is not impossible that powder magnets will soon be produced that will be superior to anything that we know of today. An important step in this direction is the recently reported Bismanol,[3] a compacted powder of manganese bismuthide which was found by Guillaud[4] to have a high crystal anisotropy and a high coercive force.

FERRITES

Now let us turn our attention to the ferrites which have recently come into commercial prominence particularly in applications where high frequency is important. These are refractory oxides composed of iron oxide, Fe_2O_3, chemically combined with one or more of the oxides of nickel, manganese, zinc, cobalt, etc. Usually they are pressed in powder form to the required shape and fired, after which they are hard, strong, and brittle.

When any magnetic material is magnetized in

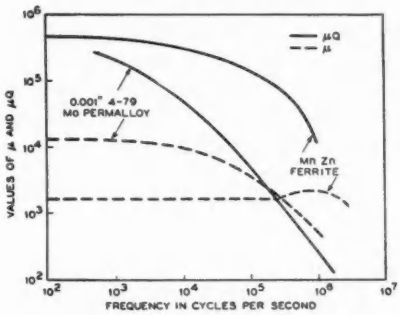


FIG. 8. Effective permeability μ, and its product with the quality factor Q (reciprocal of energy loss) as dependent on frequency. Note difference between laminated metallic material and unlaminated ceramic (ferrite), and superiority of the latter at high frequencies (1 Mc/sec).

a high frequency field, the power losses resulting from eddy currents become an increasingly important factor. Ordinarily eddy currents are reduced by laminating the material, and Fig. 8 shows the way that the losses vary in a well-laminated structure of a high quality metallic material, Molybdenum Permalloy, in which the laminations are only 0.001 in. thick. The losses are given in terms of Q (the quality factor), the reciprocal of the loss. More specifically $Q=2\pi fL/\Delta R$, L being the inductance at the frequency f, of a coil of wire containing the material, and ΔR the effective resistance of the coil caused by the presence of the material.

[3] Adams, Hubbard, and Syeles, "Bismanol, A New Permanent Magnet." Navord Report 2440, U. S. Naval Ordnance Laboratory, Silver Spring, Maryland (May 20, 1952), pp. 1–15.

[4] C. Guillaud, "Ferromagnetism of binary alloys of manganese," Thèse, l'Université de Strasbourg (France) 1943, pp. 1–129.

In addition to the increase of loss at high frequencies, (ΔR), we have a decrease in the permeability, μ, which arises because the whole flux cannot follow the rapid alternations of the field. The best single criterion for a material for use at high frequencies is the product of these quantities, μQ. The μQ curves for Molybdenum Permalloy and for manganese zinc ferrite are shown by the solid lines in Fig. 8. One sees that in 1-mil Molybdenum Permalloy the curve falls rather rapidly with increasing frequency, whereas in the curve for the ferrite (not laminated) it holds up well to over one megacycle.

The efficacy of the ferrites is due to their very high electrical resistivity; they do not require lamination. Although the resistivity of the Permalloy is about 55×10^{-6} ohm-cm, that of the nonmetallic refractory ferrites is about 10^2 to 10^6. The ratio for the two materials is then more than a million. Although the permeability of the ferrites at low frequencies is not high, their μQ is relatively large at frequencies of 1 megacycle and above. There have obviously been many applications of this material in communication.

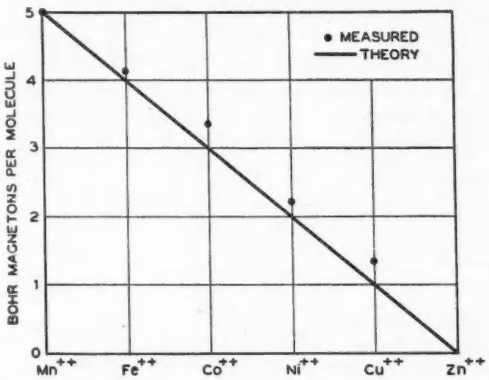


FIG. 9. Theoretical and observed atomic moments of various ferrites ($MnFe_2O_4$, etc.), showing general validity of Néel theory based on atomic moments of Table I.

We are concerned here primarily with the structure of the ferrites and its relation to their magnetic properties. The chemical formula of the nickel ferrite (for example) may be written, $NiO\cdot Fe_2O_3$ or $NiFe_2O_4$. The crystal structure of these materials is determined first of all by the oxygen atoms, which are much larger than the metal atoms, and which form a close-packed structure like that of a pile of billiard balls. In this structure there are two kinds of interstices, in which the metal atoms lie. One kind, the "*A*" position, is equidistant from four oxygen atoms (tetrahedral holes); the other, or "*B*" position is surrounded by six oxygen atoms (octahedral holes). In a normal ferrite the bivalent atoms are in the "*A*" positions and the Fe^{+++} atoms in the "*B*" positions. These ferrites are nonmagnetic; the magnetic ferrites all have the *inverse* spinel structure, with the Fe^{+++} atoms equally distributed between the "*A*" and "*B*" positions, and all of the bivalent atoms, e.g., Ni^{++} in $NiFe_2O_4$, in the "*B*" positions. This may be represented schematically as follows:

$$\begin{array}{lll} A & B & \\ Ni^{++} & (Fe^{+++}, Fe^{+++})\ O_4^{--} & \text{(normal spinel)}, \\ Fe^{+++} & (Ni^{++},\ \ Fe^{+++})\ O_4^{--} & \text{(inverse spinel)}. \\ \rightarrow & \leftarrow\quad\ \leftarrow & \end{array}$$

It was proposed by Néel[5] that in the magnetic inverse spinel the magnetic moments of the ions in the "*A*" positions are held by quantum-mechanical forces so that they are antiparallel to those in the "*B*" positions, as shown by the arrows below the symbols. The net magnetic moment of the ferrite molecule is then the mo-

TABLE I. Atomic structure and magnetic moments of some iron-group ions.

	1st Shell	2nd Shell		3rd Shell	4th Shell	Net Moment
+Spin	1	1, 3	26 Fe	1, 3, 5	1	4(2.2)[a]
−Spin	1	1, 3		1, 3, 1	1	
	1	1, 3	23 Fe^{+++}	1, 3, 5	0	5
	1	1, 3		1, 3, 0	0	
	1	1, 3	23 Mn^{++}	1, 3, 5	0	5
	1	1, 3		1, 3, 0	0	
	1	1, 3	24 Fe^{++}	1, 3, 5	0	4
	1	1, 3		1, 3, 1	0	
	1	1, 3	25 Co^{++}	1, 3, 5	0	3
	1	1, 3		1, 3, 2	0	
	1	1, 3	26 Ni^{++}	1, 3, 5	0	2
	1	1, 3		1, 3, 3	0	
	1	1, 3	27 Cu^{++}	1, 3, 5	0	1
	1	1, 3		1, 3, 4	0	
	1	1, 3	28 Zn^{++}	1, 3, 5	0	0
	1	1, 3		1, 3, 5	0	

[a] Net moment is 4 in a free atom, 2.2 in the compact metal.

[5] L. Néel, Ann. phys., series 12, 3, 137 (1948).

ment of the bivalent metal ion, i.e., of Ni^{++} in $NiFe_2O_4$.

The magnetic moments of the various ions that occur in the common ferrites are shown in Table I. These are derived from the atomic structure in much the same way as the moment of a free iron atom is derived from the structure shown in Fig. 1 and are simply the differences between the number of electrons in the third shell having plus spins and those having minus spins. In $NiFe_2O_4$ the moments of the Fe^{+++} atoms cancel each other and the net moment of the molecule is that of the Ni^{++} ions or 2 Bohr magnetons.

The atomic moments of various ferrites, as predicted by atomic theory, can be compared with experiment by measuring the saturation magnetization of the materials at low temperatures and extrapolating to 0°K. Measurements have been made[6] for manganese, iron, cobalt, nickel, and copper ferrites, and Fig. 9 compares the number of Bohr magnetons deduced from the magnetic measurements with the predictions of Néel's theory, as indicated by the straight line. The small discrepancy between the calculated and observed moments has been attributed to the magnetic moments caused by the orbital motions of the electrons, which are not taken into account by theory.

We can see then that we have a very good basic understanding of the atomic structure of the ferrites and its relation to one of the principle magnetic quantities, the magnetization at saturation.

[6] For references to the sources of data see L. Néel, Comptes rend. **230**, 190 (1950).

DOMAIN MOVIES

Further description of the processes that occur in magnetic materials, when a magnetic field is applied, are illustrated in the film "*Action Pictures of Magnetic Domains*," prepared by H. J. Williams, C. Kittel, and F. M. Tylee. A limited number of copies of this film are available for loan for educational purposes. Requests should be addressed to the Publication Department, Bell Telephone Laboratories, 463 West Street, New York 13, N. Y. The normal time for projection (without sound) is about 30 minutes.

Practical Aids for Physics Teachers

The *American Journal of Physics* plans to introduce a new feature as soon as possible under the title, "Practical Aids for Physics Teachers." At first, it may begin in a small way, but we hope that it will grow to occupy several pages in every monthly issue.

The aids that go under such a title must be contributed; they cannot be prepared in the Editor's office. Material of the following kind would be most welcome:

A. Examination papers that seem to have been successful ones at all levels of difficulty, even into the beginning graduate level, but especially at the elementary level.

B. Problems, particularly at the elementary level; also at the upper undergraduate level. If problems are very difficult, solutions should be provided. They will be printed with the problems.

C. Useful analogies that may be used to clarify some of the basic concepts of physics; or useful illustrations of fundamental ideas.

D. Neat ideas for lecture demonstrations.

E. Useful material in books, magazines or trade journals that are not likely to come to the attention of the average teacher of physics but which offer illuminating discussions of simple matters. Give the reference completely and specifically; summarize in one or two sentences.

The contributor of each item will be recognized in the following way: At the end of each item (or it may be at the beginning) will be placed the following statement: *Contributed by John Doe, Penpoint College.* This form will be used if the contributor wishes to be identified.

Should the contributor not wish to be identified, a form such as the following should be employed, for example, in the case of a contributed examination paper: *Used in a first-year physics course for liberal arts students in a private college of approximately 800 total enrollment.* Or else it might run: *Used in a physics course for sophomore engineering students in a state university of total enrollment 15 000.*

Now you see what I mean. Please send me contributions. The only stipulation that I must make is that everything must be typed double-spaced. When I say everything, I mean *everything*. Otherwise, there is no room for editorial markings. A mimeographed copy that is already double-spaced and of good quality would be acceptable.

This is your Journal, your Association. Won't you help?

THOMAS H. OSGOOD
Editor

Rocking Experiment with Two Degrees of Freedom

JOHN SATTERLY
University of Toronto, Toronto 5, Canada
(Received July 31, 1952)

A ring stands, with axis horizontal, on a horizontal table and a cylinder is placed, also with axis horizontal, resting within it. If motion in the *same* horizontal direction be given to both, they both roll and the motion is of no particular interest. If, however, motion in *opposite* directions be given to the ring and the cylinder "rocking" occurs. This motion is interesting because its theoretical solution involves the use of Lagrangian equations, there being two degrees of freedom. Theory and experiment are here given for rings (thin and thick, complete and semicircular) and cylinders (hollow and solid) of different sizes. Good agreement is observed.

I. CYLINDRICAL SHELL

THE experiments described in this paper were prompted by previous experiments on rocking[1] and by a problem set in Lamb's *Higher Mechanics*.[2] It reads,

> A thin cylindrical shell of radius R and mass M rests with its axis horizontal on a horizontal table. Inside it is placed a circular cylinder of mass m, radius r, and radius of gyration k. If the system is set rocking, form the equations of finite motion. If the motion be small, prove that the length of the equivalent simple pendulum is given by

$$L=(R-r)\frac{2M(1+k^2/r^2)}{2M+m(1+k^2/r^2)}. \tag{1}$$

We assume no slipping between the shell (or ring) and the cylinder, nor between the ring and the table. The centers of gravity of the ring and cylinder move in opposite directions.

Figure 1 illustrates the motion. In it, O, the center of gravity of the ring, is moving to the right; and P, the center of gravity of the cylinder, is climbing upwards on the left. In Fig. 1 T is the point of contact of the ring with the table and Q the point of contact of the ring with the cylinder. In the central position O_1 and P_1 are the centers of ring and cylinder, respectively, and T_1 the point of contact of ring with table and also of ring with cylinder. We see that in the motion which has ensued since the bodies last passed through the central position the point on the ring which was the point of contact of ring and table has moved up to T_2 and the point on the cylinder which was, in the central position, coincident with T_1 has now reached T_3. Since there has been no slipping, the straight line TT_1 equals the arc TT_2 on the shell and the arc QT_2 on the shell equals the arc QT_3 on the cylinder.

Denote the angles TOT_1, TOQ by α and θ, respectively. Draw PS a vertical through P. The angle QPS also equals θ. Denote the angle T_3PS by ϕ. The radius of the cylinder which was vertical in the central position is now inclined at angle ϕ to the vertical. The rotation of this radius is a measure of the rotation of the cylinder.

We have more than one degree of freedom, and the simple method used earlier by me to get the frequency of vibration will not suffice here. Lagrange's equations for "double freedom" are more convenient. These are[3]

$$\begin{aligned}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1}-\frac{\partial T}{\partial q_1}&=-\frac{\partial V}{\partial q_1},\\ \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2}-\frac{\partial T}{-\partial q_2}&=-\frac{\partial V}{\partial q_2},\end{aligned} \tag{2}$$

where q_1 and q_2 are the two variables, and T and V are the kinetic energy and potential energy, respectively, of the system.

In Fig. 1 we have designated three angles α, θ, ϕ, but a geometric relation between these serves to eliminate ϕ leaving the two variables α and θ which will take the place of q_1 and q_2 in the above statement of Lagrange's equations.

[1] John Satterly, Am. J. Phys. 18, 405 (1950) and 19, 511 (1951).
[2] H. Lamb, *Higher Mechanics* (Cambridge University Press, Cambridge, 1929), Ex. XIV, No. 6, p. 255.
[3] H. Lamb, *Dynamics* (Cambridge University Press, Cambridge, 1923), 104–106, p. 317. See also other texts.

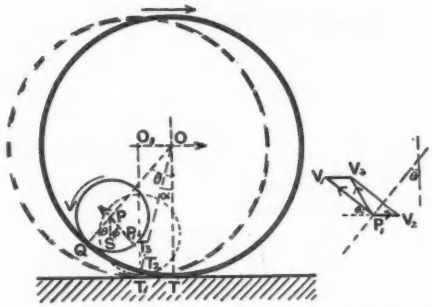


FIG. 1. A cylinder rolling and rocking within a thin circular band or ring which is itself rolling and rocking on a horizontal table. At any instant the cylinder and band are rolling in opposite directions; the figure portrays an instant when the band is moving to the right and the cylinder to the left as indicated by the arrows. The dotted circles indicate the central positions of the band and cylinder. The insert on the right is the velocity diagram for the center of the cylinder.

The geometric relation between α, θ, ϕ is obtained from the fact that since there is no slipping the length of the arc QT_2 whose radius is R equals the length of the arc QT_3 whose radius is r. Therefore

$$R(\theta-\alpha)=r(\theta+\phi),$$

or

$$\phi=\{(R-r)\theta-R\alpha\}/r. \tag{3}$$

To calculate the kinetic energy of the cylinder we need the velocity of its center. Consider the velocity diagram of P shown as an insert to Fig. 1. Relative to the center of the shell, the velocity of the center of the cylinder is $P_1V_1=(R-r)\dot{\theta}$ at right angles to OP and therefore inclined at the angle θ to the horizontal. The velocity of the center of the shell is $P_1V_2=R\dot{\alpha}$ horizontally. Therefore the velocity of the center of the cylinder with respect to the table-top (assumed to be at rest) is the vector sum P_1V_3 of P_1V_1 and P_1V_2, and by geometry

$$\begin{aligned}(P_1V_3)^2&=(P_1V_1)^2+(P_1V_2)^2-2P_1V_1\cdot P_1V_2\cos\theta\\&=\{(R-r)\dot{\theta}\}^2+(R\dot{\alpha})^2\\&\qquad-2\{(R-r)\dot{\theta}\}(R\dot{\alpha})\cos\theta.\end{aligned}$$

If we limit the vibrations to those of very small amplitude, we may replace $\cos\theta$ by unity, whence we get

$$\text{(velocity of center of cylinder)}^2=\{(R-r)\dot{\theta}-R\dot{\alpha}\}^2. \tag{4}$$

We now build up the Lagrangian equations: The kinetic energy of the shell, which is the energy of rotation about T may be written

$$T_M=\tfrac{1}{2}M(K^2+R^2)\dot{\alpha}^2,$$

where K = radius of gyration of the shell about its central axis. Since $K=R$ for a thin cylindric shell, this reduces to $T_M=MR^2\dot{\alpha}^2$.

The kinetic energy of the cylinder T_m equals the sum of the energy of rotation about its center and the energy of translation of the whole considered concentrated at its center; Therefore

$$\begin{aligned}T_m&=\tfrac{1}{2}mk^2\dot{\phi}^2+\tfrac{1}{2}m\{(R-r)\dot{\theta}-R\dot{\alpha}\}^2,\text{ by Eq. (4)},\\&=\tfrac{1}{2}m(1+k^2/r^2)\{(R-r)\dot{\theta}-R\dot{\alpha}\}^2,\text{ by Eq. (3)}.\end{aligned}$$

Therefore the whole kinetic energy T, being equal to T_M+T_m, is given by

$$T=MR^2\dot{\alpha}^2+\tfrac{1}{2}m(1+k^2/r^2)\{(R-r)\dot{\theta}-R\dot{\alpha}\}^2. \tag{5}$$

The potential energy V is given by V_M+V_m, in which V_M does not change and V_m is the weight of the cylinder times the change of elevation of its center. Therefore

$$V=-mg(R-r)\cos\theta+\text{constant}. \tag{6}$$

We now form the Lagrangian equations for α:

$$\frac{\partial T}{\partial\dot{\alpha}}=2MR^2\dot{\alpha}+m(1+k^2/r^2)\times\{(R-r)\dot{\theta}-R\dot{\alpha}\}(-R),$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial T}{\partial\dot{\alpha}}&=2MR^2\ddot{\alpha}-m(1+k^2/r^2)R\{(R-r)\ddot{\theta}-R\ddot{\alpha}\}\\&=\{2MR^2+m(1+k^2/r^2)R^2\}\ddot{\alpha}\\&\qquad-m(1+k^2/r^2)R(R-r)\ddot{\theta},\end{aligned}$$

and

$$\frac{\partial T}{\partial\alpha}=0,$$

$$\frac{\partial V}{\partial\alpha}=0.$$

Therefore the Lagrangian equation for α is

$$\{2MR^2+m(1+k^2/r^2)R^2\}\ddot{\alpha}-m(1+k^2/r^2)R(R-r)\ddot{\theta}=0, \tag{7}$$

which enables us to express $\ddot{\alpha}$ in terms of $\ddot{\theta}$.

We now form the Lagrangian equations for θ:

$$\frac{\partial T}{\partial \dot{\theta}}=0+m(1+k^2/r^2)\{(R-r)\dot{\theta}-R\dot{\alpha}\}(R-r),$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}}=m(1+k^2/r^2)\{(R-r)\ddot{\theta}-R\ddot{\alpha}\}(R-r),$$

$$\frac{\partial T}{\partial \theta}=0,$$

$$\frac{\partial V}{\partial \theta}=mg(R-r)\sin\theta=mg(R-r)\theta,$$

as θ is limited to very small values.

Therefore, the Lagrangian equation for θ is

$$-m(1+k^2/r^2)R(R-r)\ddot{\alpha}+m(1+k^2/r^2)\times(R-r)^2\ddot{\theta}+mg(R-r)\theta=0. \quad (8)$$

By use of Eq. (7) we substitute for $\ddot{\alpha}$ in Eq. (8) thus getting an equation involving θ, $\ddot{\theta}$, and constants. It is

$$-m(1+k^2/r^2)R(R-r)\left\{\frac{m(1+k^2/r^2)R(R-r)}{2MR^2+m(1+k^2/r^2)R^2}\right\}\ddot{\theta}+m(1+k^2/r^2)(R-r)^2\ddot{\theta}+mg(R-r)\theta=0,$$

which reduces to

$$(R-r)\frac{2M(1+k^2/r^2)}{2M+m(1+k^2/r^2)}\ddot{\theta}+g\theta=0, \quad (8')$$

whence we see that the motion in θ is simple harmonic and that the length of the equivalent simple pendulum is given by

$$L=(R-r)\frac{2M(1+k^2/r^2)}{2M+m(1+k^2/r^2)}, \quad (9)$$

which is Lamb's expression. The period of the vibration $=2\pi(L/g)^{\frac{1}{2}}$.

TABLE I. Details of experiments with models of form (Fig. 2). The outer body: A thin brass ring (or band or shell) of mass $M=301$ g, radius R (mean) $=10.05$ cm, thickness 0.2 cm, length 2.9 cm. In all cases and in many others not detailed here there is good agreement between theory and practice.

The inner body →	Thin-walled brass tube (Fig. 2A)	Solid brass cylinder (Fig. 2B)	Solid wood cylinder (Fig. 2C)	Steel ball-bearing (Fig. 2D)
Mass m (g)	92	1400	227	440
Radius r (cm)	2.17 (mean)	2.51	6.40	2.38
Thickness (cm)	0.1	...	...	...
Length (cm)	6.3	8.3	4.3	...
L calc from Eqs. (10) to (12) (cm)	12.1	2.52	3.50	5.30
$L^{\frac{1}{2}}$	3.48	1.59	1.87	2.30
Period, calc (sec)	0.69	0.32	0.37	0.46
No. of vibrations timed, and repetitions thereof	10 in 6.7, 6.8, 6.8, 6.9 sec.	20 in 6.7, 7.0, 6.7, 6.6, 6.7 sec.	10 in 3.7, 3.8, 3.8, 3.7 sec.	3 in 1.3 sec, 7 in 3.4 sec, 10 in 4.5 sec, etc.
Period, mean obs (sec)	0.68	0.33_+	0.37_+	0.46

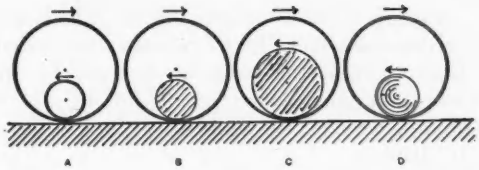


FIG. 2. Four examples of a rolling and rocking brass band containing a rolling and rocking body. In (A) the inner body is a thin-walled metal cylinder, in (B) a solid metal cylinder, in (C) a larger solid wooden cylinder and in (D) a steel sphere (a ball-bearing). Arrows indicate the direction of motion for one half the period; in the other half the directions are reversed.

Corollaries

(1) An alternative expression for L is

$$L=2(R-r)\frac{1}{2/(1+k^2/r^2)+m/M}.$$

(2) If the cylinder is hollow and thin walled, $k^2=r^2$; therefore

$$L=(R-r)\frac{4M}{2M+2m}=2(R-r)\frac{M}{M+m}=2(R-r)\frac{1}{1+m/M}. \quad (10)$$

(3) If the cylinder is solid, $k^2=\frac{1}{2}r^2$; therefore

$$L=(R-r)\frac{3M}{2M+3m/2}=2(R-r)\frac{3M}{4M+3m},$$

$$=2(R-r)\frac{M}{4M/3+m}=2(R-r)\frac{1}{4/3+m/M}. \quad (11)$$

(4) If the cylinder is replaced by a sphere of radius r, $k^2=2r^2/5$, and therefore

$$L=(R-r)\frac{2M\times 7/5}{2M+m\times 7/5}=2(R-r)\frac{7M}{10M+7m}$$

$$=2(R-r)\frac{1}{10/7+m/M}. \quad (12)$$

Sometimes one form is useful for calculation, sometimes another. In CGS units the period, calculated from the value of L given by the masses and dimensions (Eqs. (9) to (13)), is, in Toronto, very nearly equal to $\frac{1}{5}\cdot L^{\frac{1}{2}}$. The error $=0.3$ percent.

Equations (9)–(12) have been checked with many rings (thin and thick) and cylinders (hollow and solid). I quote in Table I a few typical results (see Fig. 2).

II. THE THIN OUTER RING IS REPLACED BY A THICK RING

The proof given above for the thin-walled shell may be modified to suit the case of a thick cylindric ring of inner and outer radii R_1 and R_2, respectively. Figure 3 represents this case. The letters have the same significance as in Fig. 1 except that T_2 is now on the inner surface of the thick ring. In the central position this point was the point of contact of cylinder and ring and was vertically above T_1. In this case

$$\left.\begin{aligned} &r\phi=(R_1-r)\theta-R_1\alpha,\\ &(\text{velocity of center of cylinder})^2\\ &\qquad=\{(R_1-r)\dot{\theta}-R_2\dot{\alpha}\}^2,\\ &T_M=\tfrac{1}{2}M(K^2+R_2^2)\dot{\alpha}^2,\\ &\text{and}\\ &V_m=-mg(R_1-r)\cos\theta+\text{constant},\end{aligned}\right\}\quad(13)$$

where K is the radius of gyration of the thick ring about its axis.

With these substitutions the work proceeds as above with the result that

$$L=[R_1-r]\frac{M(1+k^2/r^2)(K^2+R_2^2)+m(1+k^2/r^2)(k^2R_1^2/r^2+R_2^2)-m(k^2R_1/r^2+R_2)^2}{M(K^2+R_2^2)+m(k^2R_1^2/r^2+R_2^2)},\quad(14)$$

which, of course, reduces to Eq. (9) if we let $R_1=R_2=R$.

Experimental check: Table II, first line, shows the results of a test in which the outer body is a thick brass ring of rectangular cross section, having a radius of gyration given by $K^2=148.5$ cm^2. For the inner body a solid brass cylinder was used giving for the length of the equivalent simple pendulum $L=8.73$ cm, and therefore the calculated period equaled 0.59 sec. Observations showed 10 vibrations in 6.0, 6.0, 6.0, and 6.0 sec, whence the observed period is 0.60 sec. The agreement is good.

The rest of Table II refers to the experiments of the subsequent sections.

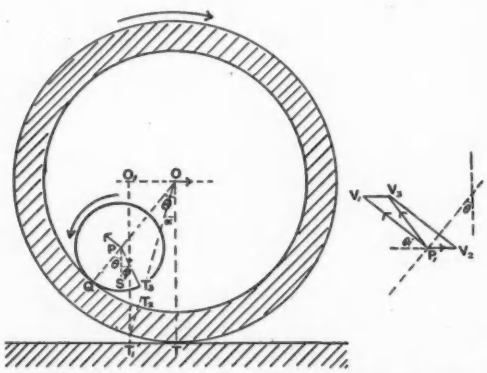


FIG. 3. A cylinder rolling and rocking within a thick circular ring which is itself rolling and rocking on a horizontal table. At any instant the cylinder and the ring are rolling in opposite directions; the figure portrays an instant when the ring is moving to the right and the cylinder to the left as indicated by the arrows. The insert on the right is the velocity diagram for the center of the cylinder.

III. THE COMPLETE RING IS REPLACED BY A THIN-WALLED SEMICIRCLE

Figure 4 shows the semicircle and the cylinder in an off-the-center position. The letters are as in Fig. 1 with the addition of G, the center of gravity of the semicircle. Point G is at a distance $2R/\pi$ from the center of the circle along its

TABLE II. Details of experiments with cylinders rocking in rings and semicircles. In all cases the objects were of brass. Observed periods of vibration were obtained by successive measurements with a stopwatch of the time required for 5, 10, or 20 vibrations.

		Motion of the two bodies in opposite directions				Motion of the two bodies in the same direction			
Outer body[a]	Inner body[a]	n (sec^{-1})	L (cm)	Period calc (sec)	Period obs (sec)	n (sec^{-1})	L (cm)	Period calc (sec)	Period obs (sec)
A	*E*		8.73	0.59	0.60				
B	*F*	16.7	3.50	0.38	0.38	3.12	101.	2.01	2.06
C	*G*	19.1	2.40	0.33	0.33	7.85	16.0	0.80	0.80
D	*E*	17.1	3.34	0.367	0.37	4.05	59.8	1.55	1.55

[a] *A* Thick ring: M—2508 g, R_1—10.65 cm, R_2—13.55 cm, thickness 1.35 cm. *B* Thin-walled semicircle: M—177.5 g, R(mean)—15.35 cm, thickness—3 mm, width—1.35 cm. *C* Thin-walled semicircle: M—115 g, R_1—4.90 cm, thickness—1.8 mm, width 5.08 cm. *D* Thick-walled semicircle: one-half of ring *A*, M—1253 g. *E* Solid cylinder: m—1400 g, r—2.51 cm, length 8.3 cm. *F* Solid cylinder: m—328 g, r—2.00 cm, length 3.05 cm. *G* Thin-walled tubing: m—41 g, r(outside)—1.20 cm, thickness—0.5 mm, length 13.6 cm.

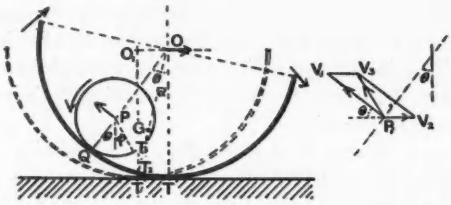


FIG. 4. A cylinder rolling and rocking within a thin-walled semicircular band which is itself rolling and rocking on a horizontal table. The figure portrays an instant when the cylinder and band are rolling in *opposite* directions as indicated by the arrows. The dotted semicircle indicates the central position of the band. The insert on the right is the velocity diagram for the center of the cylinder.

middle axis. Call this distance b_1. In the central position the moment of inertia of the semicircle about the point T_1 is equal to

$$M(R^2-b_1^2+(R-b_1)^2)=M(2R^2-2Rb_1).$$

This is also, for small observations very nearly, the moment of inertia about T. In motion the center of gravity of the semicircle rises. In the α-position the elevation is $b_1(1-\cos\alpha)$. Therefore the expressions for the energies are

$$\left.\begin{aligned} T&=\tfrac{1}{2}M(2R^2-2Rb_1)\dot{\alpha}^2 \\ &\quad+\tfrac{1}{2}m(1+k^2/r^2)\{(R-r)\dot{\theta}-R\dot{\alpha}\}^2, \\ V&=-Mgb_1\cos\alpha \\ &\quad-mg(R-r)\cos\theta+\text{constant}, \end{aligned}\right\} \quad (14a)$$

and the Lagrangian equation for α is

$$\begin{aligned} &\{M(2R(R-b_1))+m(1+k^2/r^2)R^2\}\ddot{\alpha} \\ &\quad-m(1+k^2/r^2)R(R-r)\ddot{\theta}+Mgb_1\alpha=0, \end{aligned} \quad (15)$$

and for θ is

$$\begin{aligned} &-m(1+k^2/r^2)R(R-r)\ddot{\alpha} \\ &\quad+m(1+k^2/r^2)(R-r)^2\ddot{\theta}+mg(R-r)\theta=0. \end{aligned} \quad (16)$$

The presence of the term $Mgb_1\alpha$ in Eq. (15) blocks our previous method of expressing $\ddot{\alpha}$ in terms of $\ddot{\theta}$ and substituting in Eq. (16). We now have to use the method for small oscillations about equilibrium given in Lamb's *Dynamics*, Sec. 109. Here he writes our Lagrangian equations in the form

$$\left.\begin{aligned} A\ddot{\alpha}+H\ddot{\theta}+a\alpha+h\theta&=0, \\ H\ddot{\alpha}+B\ddot{\theta}+h\alpha+b\theta&=0. \end{aligned}\right\} \quad (17)$$

To solve Eqs. (17) assume

$$\alpha=C\cos(nt+\epsilon) \text{ and } \theta=\lambda C\cos(nt+\epsilon), \quad (18)$$

where C, λ, and ϵ are constants. The equations are satisfied provided

$$\left.\begin{aligned} (n^2A-a)+(n^2H-h)\lambda&=0, \\ (n^2H-h)+(n^2B-b)\lambda&=0. \end{aligned}\right\} \quad (19)$$

Eliminating λ we have

$$\begin{vmatrix} n^2A-a & n^2H-h \\ n^2H-h & n^2B-b \end{vmatrix}=0, \quad (20)$$

which is a quadratic in n^2. The formulas (18) therefore constitute a solution of the Eqs. (17), provided n satisfies Eq. (20) and provided the coefficient λ be determined in accordance with either of the Eqs. (19). Obviously the length of the equivalent simple pendulum is given by $L=g/n^2$, and the period $=2\pi/n$.

This method of solution was applied to the case of the rocking of a large thin brass semicircle B carrying a solid brass cylinder F. When the two bodies are rocking in opposite directions observations give, for the observed period, 0.38 sec.

The values of Lamb's coefficients are

$$\begin{aligned} A&=2MR(R-b_1)+m(1+k^2/r^2)R^2 \\ &\qquad=14.64\times10^4 \text{ g-cm}^2, \\ H&=-m(1+k^2/r^2)R(R-r)=-10.08\times10^4 \text{ g-cm}^2, \\ B&=m(1+k^2/r^2)(R-r)^2=8.77\times10^4 \text{ g-cm}^2, \\ a&=Mgb_1=170.1\times10^4 \text{ dyne-cm}, \\ b&=mg(R-r)=429.3\times10^4 \text{ dyne-cm}, \\ h&=0. \end{aligned}$$

The quadratic equation in n^2 is therefore

$$(14.64n^2-170.1)(8.77n^2-429.3)=(10.08n^2-0)^2,$$

whence

$$n=16.7 \text{ or } 3.12.$$

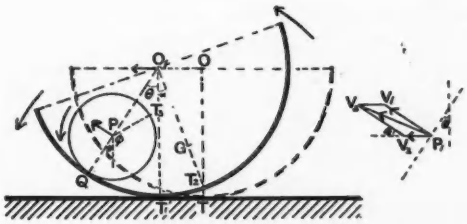


FIG. 5. A cylinder rolling and rocking within a thin-walled semicircular band which is itself rolling and rocking on a horizontal table. The figure portrays an instant in the motion when cylinder and band are rolling in the *same* direction as indicated by the arrows. The dotted semicircle shows the central position of the band. The insert on the right is a velocity diagram for the center of the cylinder at the instant portrayed.

Taking the first value of n, the calculated period, $=2\pi/16.7=0.376=0.38$ sec gives good agreement with the observed value.

But what does the value $n=3.12$ signify? Let us see what happens if instead of the cylinder and the semicircle rocking in *opposite* directions they rock in the *same* direction. In this case (see Fig. 5) the following geometrical equations hold instead of Eqs. (3) and (4).

For the equality of arc lengths, QT_2 and QT_3

$$R(\alpha+\theta)=r(\theta+\phi),$$

whence

$$r\phi=(R-r)\theta+R\alpha, \tag{21}$$

and for the velocity of the center of the cylinder (see insert to Fig. 5).

$$P_1V_3^2=P_1V_1^2+P_1V_2^2+2P_1V_1\cdot P_1V_2\cos\theta$$
$$=\{(R-r)\dot{\theta}+R\dot{\alpha}\}^2, \tag{22}$$

when θ is very small.

For this case the Lagrangian equations become

$$\left.\begin{aligned}&\{2MR(R-b_1)+m(1+k^2/r^2)R^2\}\ddot{\alpha}\\&\quad+m(1+k^2/r^2)R(R-r)\ddot{\theta}+Mgb_1\alpha=0,\\&m(1+k^2/r^2)R(R-r)\ddot{\alpha}\\&\quad+m(1+k^2/r^2)(R-r)^2\ddot{\theta}+mg(R-r)\theta=0.\end{aligned}\right\} \tag{23}$$

The only difference from those of Eqs. (15), (16), and (17) is that Lamb's coefficient H is now positive.

Therefore, it follows that the quadratic in n^2 is now the same as before and that the $n=3.12$ corresponds to the present case of the two bodies rocking in the *same* direction at the *same* time. From $n=3.12$ the calculated period is $2\pi/3.12=2.01$ sec.

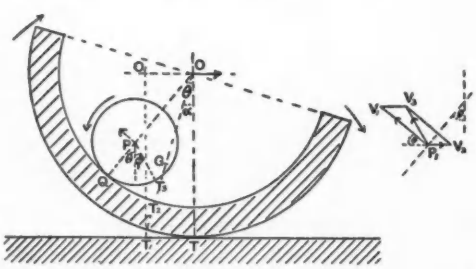


FIG. 6. A cylinder rolling and rocking within a thick walled semicircular ring which is itself rolling and rocking on a horizontal table. This figure portrays the case when the cylinder and semicircle are, at the same instant, moving in *opposite* directions, as indicated by the arrows. The insert on the right is the velocity diagram for the center of the cylinder.

A fair agreement between experiment and theory was observed when the last quoted bodies, rocking in the same direction, gave an observed period 2.06 sec. Of course, the semicircle is not theoretically thin.

On a second trial a smaller, wider semicircle was taken, and the cylinder was a piece of *thin* brass tubing. These objects are designated as C and G in Table II, where dimensions and masses are tabulated.

Lamb's coefficients are $A=4137$ g-cm^2, $H=-1558$ g-cm^2, $B=1181$ g-cm^2, $a=366g$ dyne-cm, $b=156g$ dyne-cm, and $h=0$. In coefficients a and b the acceleration of gravity is represented by g. The quadratic equation in n^2 is, therefore,

$$(n^2 4137-366g)(n^2 1181-156g)=\{n^2(-1558)\}^2$$

or

$$n^4\times4.89\times10^6-n^2\times10.80\times10^8$$
$$+5.48\times10^{10}=n^4\times2.43\times10^6,$$

whence $n=19.5$ and 7.65 and the calculated periods 0.32 sec and 0.82 sec. The observations were (1) when rocking in the same direction, an observed period of 0.80 sec, and (2) when rocking in opposite directions; an observed period of 0.33 sec. The agreement is good.

IV. THE THIN-WALLED SEMICIRCLE IS REPLACED BY A THICK-WALLED SEMICIRCLE

But no ring or semicircle is as thin as a theoretical thin ring or semicircle. If it were so it would not keep its shape. So let us now take a thick semicircle and see how the equations are modified.

Let the internal and external radii by R_1 and R_2 (Fig. 6). Let the center of gravity G of the semicircle be at a distance b_1 from the center of curvature O. By geometry

$$b_1=(4/3\pi)(R_2^3-R_1^3)/(R_2^2-R_1^2).$$

(1) Rocking in Opposite Directions

As before (see Eq. 13 for the thick circle) we have

$$r\phi=(R_1-r)\theta-R_1\alpha$$

and (velocity of center of cylinder)^2 =

$$\{(R_1-r)\dot{\theta}-R_2\dot{\alpha}\}^2;$$

also

$$T_M=\tfrac{1}{2}M\{K^2+(R_2-b_1)^2\},$$

$$T_m=\tfrac{1}{2}mk^2\left\{\frac{(R_1-r)\dot{\theta}-R_1\dot{\alpha}}{r}\right\}^2+\tfrac{1}{2}m\{(R_1-r)\dot{\theta}-R_2\dot{\alpha}\}^2,$$

and

$$V=-Mgb_1\cos\alpha-mg(R_1-r)\cos\theta+\text{constant},$$

and the Lagrangian equations are for α

$$\left.\begin{aligned}&[M\{K^2+(R_2-b_1)^2\}+m(k^2/r^2R_1^2+R_2^2)]\ddot{\alpha}\\&\quad-m(R_1-r)(k^2/r^2R_1+R_2)\ddot{\theta}+Mgb_1\alpha=0\\&\text{and for }\theta\\&-m(R_1-r)(k^2/r^2R_1+R_2)\ddot{\alpha}+m(R_1-r)^2\\&\qquad\times(1+k^2/r^2)\ddot{\theta}+mg(R_1-r)\theta=0.\end{aligned}\right\}\quad(24)$$

Experiment: The thick semicircle D used was one-half of the thick ring A used in Sec. II. The cylinder was also the same as in Sec. II:

$$b_1=7.74\text{ cm (calc)},$$
$$K^2=\tfrac{1}{2}(R_1^2+R_2^2)-b_1^2=88.6\text{ cm}^2,$$
$$R_1-r=8.14\text{ cm},\quad (R_1-r)^2=66.3\text{ cm}^2,$$
$$R_2-b_1=5.81\text{ cm},\quad (R_2-b_1)^2=33.8\text{ cm}^2.$$

The coefficients in Lamb's equations (17) are

$$A=M\{K^2+(R_2-b_1)^2\}+m(k^2/r^2R_1^2+R_2^2)=4.90\times10^5\text{ g-cm}^2,$$
$$H=-m(R_1-r)(k^2/r^2R_1+R_2)=-2.15\times10^5\text{ g-cm}^2,$$
$$B=m(R_1-r)^2(1+k^2/r^2)=1.39\times10^5\text{ g-cm}^2,$$
$$a=Mgb_1=95.0\times10^5\text{ dyne-cm},$$
$$b=mg(R_1-r)=111.7\times10^5\text{ dyne-cm},$$
$$h=0.$$

The equation for n^2 is

$$(4.90n^2-95.0)(1.39n^2-111.7)=(-2.15n^2-0)^2,$$

whence $n=17.1$ or 4.05. The calculated periods are 0.367 and 1.55 sec, respectively.

Observations: For semicircle and cylinder rocking and moving in opposite directions at the same time the observed period was 0.37 sec.

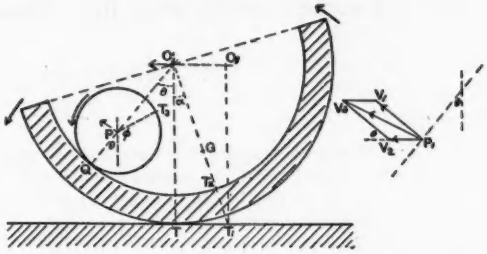


FIG. 7. A cylinder rolling and rocking within a thick-walled semicircular ring which is itself rolling and rocking on a horizontal table. The figure portrays the case when, at the same instant, the cylinder and the semicircle are moving in the *same* direction as indicated by the arrows. The insert on the right is the velocity diagram for the center of the cylinder.

(2) Rocking in the Same Direction

For the bodies rocking and moving in the same direction at the same time (Fig. 7) the equations follow the same pattern as in the case of the thin semicircle. The mean observed period was 1.55 sec.

When moving in the same direction it was observed that a jiggling motion corresponding to the mode of greater frequency tends to grow and superpose itself on the mode of greater period. The existence of two modes of rocking with the semicircle makes its problem a little more interesting than that of the complete ring. Both, however, are good fun. The lulling effect of the rocking cradle on the loosely packed-in baby provides another example of happiness easily obtained in this way. These experiments give a student good exercise in the use of the Lagrangian equations in a problem of two degrees of freedom. The materials for experimentation are very simple and are available in any good workshop. A good stopwatch is, however, necessary. I use one where the complete revolution of the seconds hand takes 10 seconds.

I have much pleasure in thanking a colleague, Dr. D. G. Ivey, for directing my attention to Lamb's problem and for help in the first part of the theory.

Geometrical Solution for a Nuclear Moment in a Magnetic Field*

ROALD K. WANGSNESS
U. S. Naval Ordnance Laboratory, White Oak, Maryland
(Received October 3, 1952)

The general solution of the classical equation of motion $d\boldsymbol{\mu}/dt = g(\boldsymbol{\mu}\times\mathbf{H})$, when $\mathbf{H}$ consists of a constant field plus a perpendicular rotating component can easily be obtained after transforming the problem to a properly chosen rotating coordinate system. The value of this method for teaching purposes is discussed.

IN a recent article in this journal, Archibald[1] pointed out that the behavior of an otherwise free nuclear magnetic moment $\boldsymbol{\mu}$ in an external magnetic field $\mathbf{H}$ provides a useful source of classroom problems, particularly for a class in quantum mechanics. The solution of the classical equations of motion for the fields of greatest practical importance is also of interest in that it proves to be a good nontrivial example of the use of the general transformation of time derivatives into rotating coordinate systems which students otherwise tend to feel is useful only for the derivation of Euler's equations for the motion of a rigid body. As we shall see, this elegant method, which was first suggested by H. C. Torrey, enables us to write down the general solution of the vector equation of motion almost at a glance.

The classical equation of motion, torque equals rate of change of angular momentum, becomes[1]

$$d\boldsymbol{\mu}/dt = g(\boldsymbol{\mu}\times\mathbf{H}), \tag{1}$$

where g, the gyromagnetic ratio, is the ratio of the magnetic moment to the nuclear angular momentum. If $\mathbf{H}$ is constant, the solution of Eq. (1) is a steady precession about the direction of $\mathbf{H}$ with $\boldsymbol{\mu}$ making a constant angle to this direction.[2] The (circular) frequency of precession is the Larmor frequency

$$\omega_L = g|\mathbf{H}|. \tag{2}$$

For simplicity, we shall assume that g is positive, so that the projection of $\boldsymbol{\mu}$ on a plane perpendicular to $\mathbf{H}$ rotates clockwise when viewed in a direction opposite to that of $\mathbf{H}$. If g is negative, the sense of precession is reversed.

In resonance experiments, $\mathbf{H}$ is made to depend on the time and generally consists of a constant component H_0 and a circularly polarized component rotating in a plane perpendicular to H_0 and in the same sense as the Larmor precession. The components of $\mathbf{H}$ can therefore be written as

$$H_x = H_1\cos\omega t, \quad H_y = -H_1\sin\omega t, \quad H_z = H_0. \tag{3}$$

If we let $D\boldsymbol{\mu}/Dt$ be the rate of change of $\boldsymbol{\mu}$ as seen in a coordinate system rotating with angular velocity $\boldsymbol{\omega}_c$ with respect to the laboratory system in which Eq. (3) is applicable, then we have[3]

$$d\boldsymbol{\mu}/dt = (D\boldsymbol{\mu}/Dt) + (\boldsymbol{\omega}_c\times\boldsymbol{\mu}). \tag{4}$$

Combining Eqs. (1) and (4), we find that

$$D\boldsymbol{\mu}/Dt = g\{\boldsymbol{\mu}\times[\mathbf{H} + (\boldsymbol{\omega}_c/g)]\}. \tag{5}$$

Thus, in the rotating system, the behavior of $\boldsymbol{\mu}$ is determined by an equation of the same form as Eq. (1) with an "effective" field in this system given by $\mathbf{H} + (\boldsymbol{\omega}_c/g)$. The advantage of this formulation is that by making a judicious choice of the rotating system we can give the effective field such a simple form that the solution of Eq. (5) is very easy to obtain.

Our first transformation is to a primed system whose x' axis lies along $\mathbf{H}_1$ and rotates with it as illustrated in Fig. 1(a). This means that we set $\boldsymbol{\omega}' = -\omega\mathbf{k}$, so that we find from Eq. (5) the field in this system to have the components

$$H_{x'} = H_1, \quad H_{y'} = 0, \quad H_{z'} = H_0 - (\omega/g). \tag{6}$$

If we introduce the abbreviations

$$\Delta = gH_0 - \omega = \omega_0 - \omega, \quad \omega_1 = gH_1,$$

* Supported in part by the U. S. Office of Naval Research.
[1] W. J. Archibald, Am. J. Phys. 20, 368 (1952).
[2] G. E. Pake, Am. J. Phys. 18, 438 (1950).
[3] G. Joos, *Theoretical Physics* (Blackie and Son, Limited, London, 1934), p. 144; H. Goldstein, *Classical Mechanics* (Addison-Wesley Press, Inc., Cambridge, Massachusetts, 1950), p. 132.

the components in the primed system can be written as

$$H_{x'}=\omega_1/g,\quad H_{y'}=0,\quad H_{z'}=\Delta/g.$$

These are indicated in Fig. 1(b).

We also see from Fig. 1(b) that the resultant field in the primed system lies in the $x'z'$ plane, makes an angle θ with the z' axis, and has the *constant* magnitude α/g, where

$$\alpha=[\Delta^2+\omega_1^2]^{\frac{1}{2}},$$

and

$$\cos\theta=\Delta/\alpha,\quad \sin\theta=\omega_1/\alpha.$$

Next, we transform to a doubly primed system by a simple rotation of axes through an angle θ about the y' axis so that the z'' axis is parallel to the resultant field α/g. The components of the field in this system are then

$$H_{x''}=H_{y''}=0,\quad H_{z''}=\alpha/g. \tag{7}$$

This system is also shown in Fig. 1(b).

Since the field in the doubly primed system is constant, the motion of $\boldsymbol{\mu}$ is a simple precession about the z'' axis while making a constant angle ϕ with this axis. According to Eqs. (2) and (7), the frequency is α. Therefore, if we take into account the sense of precession as previously discussed, the general solution in this system is

$$\left.\begin{aligned}\mu_{x''}&=\mu\sin\phi\cos(\alpha t-\epsilon)\\ \mu_{y''}&=-\mu\sin\phi\sin(\alpha t-\epsilon)\\ \mu_{z''}&=\mu\cos\phi=\text{const},\end{aligned}\right\} \tag{8}$$

where $\mu=|\boldsymbol{\mu}|$ and ϕ and ϵ are constants giving the orientation of $\boldsymbol{\mu}$ at $t=0$.

We need not have stopped with this last transformation, of course, but could have made one more to a system rotating about z'' with angular velocity $\boldsymbol{\omega}'''=-\alpha\mathbf{k}''$. The resultant field is then zero by Eq. (5), so that $\boldsymbol{\mu}$ is a constant in this new system. Transforming back to x'', y'', z'' axes, we then get Eqs. (8).

To obtain the general solution in the laboratory coordinates, we see that we need only use the equations below which are easily obtained by inspection of Fig. 1:

$$\begin{aligned}\mu_{x'}&=\mu_{x''}\cos\theta+\mu_{z''}\sin\theta\\ \mu_{y'}&=\mu_{y''}\\ \mu_{z'}&=-\mu_{x''}\sin\theta+\mu_{z''}\cos\theta\end{aligned}$$

and

$$\begin{aligned}\mu_x&=\mu_{x'}\cos\omega t+\mu_{y'}\sin\omega t\\ \mu_y&=-\mu_{x'}\sin\omega t+\mu_{y'}\cos\omega t\\ \mu_z&=\mu_{z'}.\end{aligned}$$

Finally, combining these equations with Eq. (8), we have

$$\left.\begin{aligned}\mu_x&=\mu\{[\cos\phi\sin\theta+\sin\phi\cos\theta\cos(\alpha t-\epsilon)]\\ &\qquad\times\cos\omega t-\sin\phi\sin(\alpha t-\epsilon)\sin\omega t\}\\ \mu_y&=\mu\{[\cos\phi\sin\theta+\sin\phi\cos\theta\cos(\alpha t-\epsilon)]\\ &\qquad\times\sin\omega t+\sin\phi\sin(\alpha t-\epsilon)\cos\omega t\}\\ \mu_z&=\mu[\cos\phi\cos\theta-\sin\phi\sin\theta\cos(\alpha t-\epsilon)].\end{aligned}\right\} \tag{9}$$

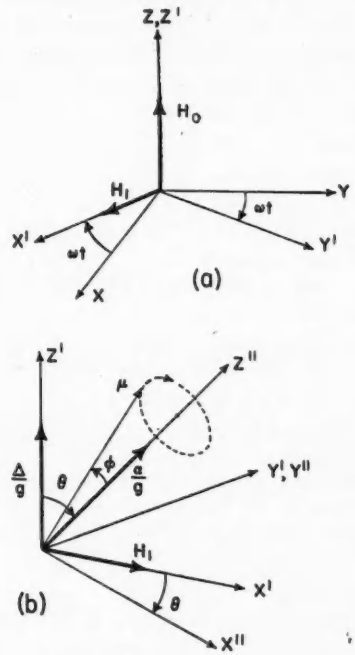


FIG. 1. The coordinate systems used to simplify the equations of motion. The heavy arrows are the magnetic field components in the different systems.

Equation (9) is actually the desired general solution since it contains two constants which are sufficient to specify the initial orientation of $\boldsymbol{\mu}$ and, as can easily be verified by substitution, satisfies Eq. (1) with the field components as given by Eq. (3).

The general motion is thus seen to be a pre-

cession of frequency ω with a superimposed nutation of frequency α.

Explicit expressions for two cases of practical importance, which were also discussed by Archibald, can easily be obtained from Eq. (9).

First, suppose that at $t=0$, $\mu_z=\mu$, and $\mu_x=\mu_y=0$. We can see from Fig. 1(b), or verify from Eq. (9), that this determines the constants to be $\phi=\theta$ and $\epsilon=\pm\pi$. Equations (9) give at once,

$$\mu_x=\mu\sin\theta[\cos\theta(1-\cos\alpha t)\cos\omega t+\sin\alpha t\sin\omega t],$$
$$\mu_y=-\mu\sin\theta[\cos\theta(1-\cos\alpha t)\sin\omega t-\sin\alpha t\cos\omega t],$$
$$\mu_z=\mu(\cos^2\theta+\sin^2\theta\cos\alpha t).$$

These agree with Archibald's equations after one corrects his Eqs. (11) by dividing them by α and also noting that his Δ is the negative of ours.

The usual equations which are used to discuss resonance experiments correspond to the special case $\mu_z=\text{const}$ and therefore $\phi=0$, which leads to

$$\mu_x=\mu(\omega_1/\alpha)\cos\omega t,\quad \mu_y=-\mu(\omega_1/\alpha)\sin\omega t,$$
$$\mu_z=\mu(\Delta/\alpha),$$

for simple precesssion without nutation.

Lest this procedure seem too much like magic to the students, it will be instructive to have them start with the three simultaneous equations obtained from Eqs. (1) and (3), and make the analytical substitutions required by the successive transformations. At each step, one can inspect the resulting equations and verify that they do have the form of Eq. (1) but with the various field components Eqs. (6) and (7). At the same time, one sees how each step simplifies the equations until the final transformation puts them into the usual harmonic oscillator form, which is immediately soluble.

The Tape Recordings of Important Speeches

BY

E. U. CONDON, K. K. DARROW, E. FERMI, J. C. SLATER

Free tape recordings are available on loan to members of the Association who wish to use them for presentation to classes, seminars, or science clubs. The subjects are four of the six lectures presented as a Symposium on Physics Today at the Twentieth Anniversary meeting of the American Institute of Physics, Chicago, Illinois, on October 25, 1951, as follows:

The atom. E. U. CONDON.

Physics as science and art. K. K. DARROW.

The nucleus. E. FERMI.

The solid state. J. C. SLATER.

Recordings are now available in two forms:

1. *Double track.* Each speech is complete on one reel, which must be played at $3\frac{3}{4}$ inches per sec. No rewinding is necessary. Track No. 1 is played to the end, then the reels are reversed (and turned over) to play track No. 2. This procedure rewinds the tape into its original condition.
2. *Single track.* Two 7-inch reels are required for each address. They must be played at $7\frac{1}{2}$ inches per sec, and, of course, rewound to restore them to their original condition.

Requests for recordings should be made to THOMAS H. OSGOOD, Editor, *American Journal of Physics*, Michigan State College, East Lansing, Michigan. To allow some flexibility in scheduling, a request should specify a preferred date and two alternative acceptable dates, as well as the type of recording (single or double track, or either).

No charge is made for the loan of recordings. The borrower is expected, however, to pay the return postage to the Editor's office or to any other address the Editor specifies. Transportation is to be by first class mail, special delivery, which will cost less than one dollar per reel at present rates. Reels will be mailed in convenient reusable boxes.

Prospective borrowers may consult the published papers in *Physics Today*, November, 1951, January, 1952, and March, 1952.

Physics and Physiology in Diving Decompression

HILBERT SCHENCK, JR.
Pratt and Whitney Aircraft, East Hartford, Connecticut
(Received September 12, 1952)

The fundamental problem of not exceeding the critical saturation of nitrogen in a diver's body requires both physics and physiology in its solution. The basic differential equations of nitrogen saturation and desaturation in a specific tissue are integrated over an idealized dive. Constants determined through physiological research can be substituted and a time-depth ascent curve for any tissue established. A stepwise method is outlined whereby critical tissues at each portion of the ascent can fix the proper ascent curve for the entire body. The advantage of pure oxygen breathing is demonstrated numerically and the failure of helium-oxygen mixtures to shorten ascents is explained by use of the derived equations.

INTRODUCTION

ALTHOUGH the general subject of deep water diving has already received attention in these pages,[1] a fuller analysis of nitrogen saturation in organic tissue may prove of interest to physicists. This appears to be primarily a physiological problem, but the solution given here is so closely related in method of attack to certain physical situations that the boundaries between the two fields become difficult to define. It should be noted, however, that whereas a physical theory can be proved a hundred times over, a physiological theory is never rigorously applicable to all individuals, but only to the few "average" individuals in the group.

When a man breathes air under pressures greater than atmospheric, his body tissues gradually take up additional nitrogen in simple solution, the amount depending on the partial pressure of the inhaled nitrogen and the time spent in the high pressure region. The oxygen component of air is not dissolved in tissue to any extent since it combines chemically within the body. Following prolonged exposure to high pressures, a sudden return to atmospheric conditions results in supercritical nitrogen saturation in the diver's body and the formation of nitrogen bubbles. The bubbles produce a variety of symptoms including itching, pain in the joints, and even paralysis and death.

Although the cause of this disorder (called the *bends* or *caisson disease*) was known as early as 1854, and an investigation accorded it by Paul Bert in 1878, it was not until 1908 that Dr. J. S. Haldane produced the first *Decompression Tables*, regulating the ascent of a diver from the sea bottom.[2] So excellent was Haldane's work that the tables currently in use by the United States Navy[3] are almost identical to those original ones worked out almost fifty years ago.

THE DESCENT

The manner in which a specific body tissue becomes saturated with nitrogen can best be illustrated by the simple model in Fig. 1. Blood entering the lung alveoli is almost instantly saturated with nitrogen at the diver's external partial nitrogen pressure. As the blood flows through a tissue saturated with nitrogen at some lesser pressure, some of the gas will diffuse into the tissue. The nitrogen pressure rise in the tissue will be dependant on the pressure difference $(P_e - P)$, the area of contact between blood and tissue A, and the tissue resistance to nitrogen transfer R. The process can be described by the

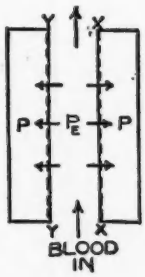


FIG. 1. Model of nitrogen saturation. Blood flows between blocks of tissue containing dissolved nitrogen at pressure P_e. Nitrogen seeps through membranes $X-X$ and $Y-Y$ into tissue with nitrogen pressure P. Rate of seepage depends on area of membrane and resistance to nitrogen movement as well as pressure differential.

[1] L. E. Dodd, Am. J. Phys. 8, 181 (1940).

[2] See J. S. Haldane, *Respiration* (Yale University Press, New Haven, 1922), pp. 334–357.

[3] Bureau of Ships Manual, Chapter 94, Section II, "Diving" (1951).

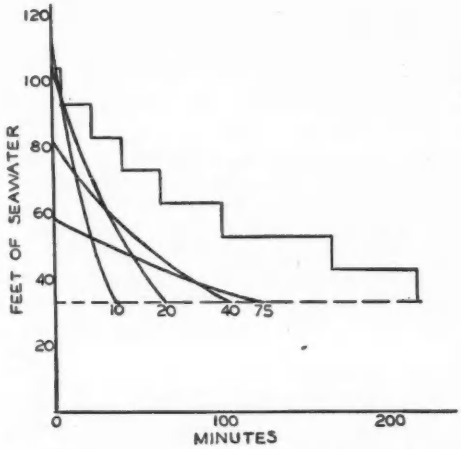


FIG. 2. Safe ascent curves for four representative tissues with indicated $T_{\frac{1}{2}}$ values for one-hour dive at 200 feet. Dotted line is sea-level pressure and stepped curve is standard *decompression* schedule.

relation,

$$\frac{dP}{dT}=\frac{A}{R}(P_e-P), \tag{1}$$

where T is the time in minutes. (The similarity between Eq. (1) and the conduction rate equation for heat transfer need not be belabored.)

If we postulate the simple case of a diver whose tissues are fully saturated with nitrogen at one atmosphere (P_{at}), suddenly being subjected to a higher, "sea bottom" pressure (P_e), Eq. (1) may be integrated to give

$$P=P_e-\left[\frac{P_e-P_{at}}{e^{AT/R}}\right], \tag{2}$$

which expresses the tissue nitrogen pressure as a function of T, the time spent "on the bottom."

It is convenient to express A/R in terms of the following equation

$$\int_0^{P_e/2}\frac{dP}{(P_e-P)}=\frac{A}{R}\int_0^{T_{\frac{1}{2}}}dT, \tag{3}$$

which states that when a given tissue, completely free of nitrogen, is subjected to a sudden pressure increase (P_e), it will require $T_{\frac{1}{2}}$ minutes to become *half-saturated* with nitrogen. Integrating and rearranging Eq. (3) results in

$$A/R=\log_e 2/T_{\frac{1}{2}}, \tag{4}$$

so that the *half-saturation time* $T_{\frac{1}{2}}$ of a tissue completely specifies its behavior providing $T_{\frac{1}{2}}$ has been determined in accordance with Eq. (3). (Note the similarity between $T_{\frac{1}{2}}$ as regards saturation and the *half-life* of an element as regards radioactive disintegration.)

THE ASCENT

Having remained a certain length of time on the ocean bottom, the diver must be brought to the surface in such a manner that the excess nitrogen held in the tissues will not form bubbles but will diffuse smoothly into the blood and be exhaled from the lungs. Following the reasoning in the previous section it is easily seen that the desaturation equation is

$$-dP'/dT'=A/R(P'-P_e'), \tag{5}$$

where P' is the nitrogen partial pressure in a specific tissue, dT' a time increment during the ascent, and P_e' the external nitrogen partial pressure at the diver's depth. (In all formulas, the prime symbol refers to ascent conditions.) We desire to regulate the ascent such that $(P'-P_e')$ is a safe maximum, and the question arises; how should P' and P_e' be related?

Through various experiments on men and animals Haldane[2] concluded that a tissue exposed to air at a given total pressure could immediately be placed in air at slightly less than half the original pressure without bubble formation. When air of atmospheric composition is used throughout the dive this results in the relation,

$$P'=nP_e', \tag{6}$$

where n for air is slightly greater than 2. Combining Eqs. (5) and (6) and integrating over the limits P to P' and 0 to T' yields the desaturation curve for any specific tissue,

$$T'=-\left(\frac{n}{n-1}\right)\frac{R}{A}\left[\log_e\left(\frac{nP_e'}{P}\right)\right], \tag{7}$$

where P is found from Eq. (2).

Although the previous equations involve nitrogen partial pressures, all P's can equally well represent total air pressures, since the nitrogen component remains a constant percentage of the total throughout the dive. Thus P_e' will be the absolute pressure surrounding the diver, and its variation with time is the *decompression curve* for the tissue considered. Now Eqs. (2) and (4) can be substituted in Eq. (7) to

give the final result,

$$T'=\frac{T_{\frac{1}{2}}}{\log_e 2}\left(\frac{n}{n-1}\right)\times\left[\log_e\left(P_e-\frac{P_e-P_{at}}{2^{T/T_{\frac{1}{2}}}}\right)-\log_e(nP_e')\right]. \quad (8)$$

Equation (8) will give the safe desaturation time T' from a bottom total pressure P_e to any lesser pressure P_e' for a tissue characterised by its *half-saturation time* $T_{\frac{1}{2}}$ providing, of course, that the T', P_e' schedule specified by Eq. (8) is followed throughout. Equation (8) assumes that descent is instantaneous as is the initial stage of the ascent when T' is negative.

Haldane[2] found that the "slowest" tissue in the body has a $T_{\frac{1}{2}}$ of seventy-five minutes and all other tissues had varying *half-saturation times* between this maximum value and a few minutes.

As an example, consider a sixty-minute dive in 200 feet of water. All pressures can be substituted as depths in sea water (33 feet equals one atmosphere) and all pressures, as noted previously, can be total pressures. Then; $P_e=233$ ft, $P_{at}=33$ ft, $T=$sixty minutes, and $n\cong 2.00$ (normal air). Time-depth ascent curves are plotted in Fig. 2 for tissues having seventy-five, forty, twenty, and ten-minute $T_{\frac{1}{2}}$ values. Each curve is the safe *decompression* schedule for the individual tissue, and the stepped curve is the *stage decompression* schedule recommended by the United States Navy[3] for this dive.

Obviously, one tissue cannot specify the entire ascent. Obeying the seventy-five minute "slow" tissue will result in an initial decrease that is dangerously large for the "faster" tissues which are more highly saturated. Following the "faster" tissue curves will provide insufficient time for the "slow" tissues to desaturate and trouble will occur at the surface. The over-all *decompression* curve must be calculated basing the early part of the ascent on the "fast" tissues and the portions nearer the surface on the "slow" tissues. Since the "slow" tissues are not under a large pressure differential at the beginning of the ascent, they lose little nitrogen until the "fast" tissues have been partially cleared and the diver has ascended to a lower pressure. The over-all time from bottom to surface is thus longer than that predicted for a specific tissue and Eq. (8) is not suitable for a rigorous analysis of the ascent. The ascent differential equation (5) must be solved in a tedious series of stepwise calculations in which several representative tissues are carried along. Equation (8) will yield the full curve in one special case, namely the condition of complete saturation of all tissues. For this situation, $T\cong\infty$, $T_{\frac{1}{2}}=$seventy-five minutes so that $P=P_e$. (Needless to say the actual, full curve is smooth and not in "stages." The problem comes in training the diver's tender to pull up his man at a logarithmic rate.)

PURE OXYGEN BREATHING

Long periods of time spent in deep water require prohibitively long ascents. To alleviate this condition, Sir Robert Davis devised a chamber to be held in sixty feet of water into which an ascending diver climbed, undressed, and relaxed while breathing pure oxygen. Bubble formation is dependent on gross outside pressure regardless of gas composition while nitrogen escape depends on external nitrogen pressure alone. Thus breathing pure oxygen has the effect of reducing P_e' in Eq. (5) to zero, but has no effect whatsoever on bubble formation. Then Eq. (5) can be integrated for this case (defining A/R as in Eq. (4)) to

$$T''=\frac{T_{\frac{1}{2}}}{\log_e 2}\left[\log_e\left(\frac{P'}{P''}\right)\right], \quad (9)$$

where P' is the tissue pressure when pure oxygen is first admitted, P'' is the tissue nitrogen pressure at any later time T''. ($T''=0$ when $P''=P'$, the instant that normal air is changed to oxygen.) Unfortunately, pure oxygen is a poison at pres-

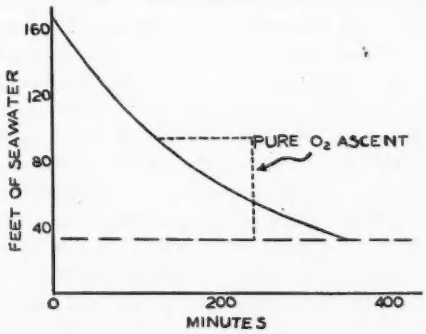


FIG. 3. Safe ascent curve for fully saturated seventy-five minute tissue from 300 ft. 33-ft line is sea level, and pure O_2 line is alternative ascent with pure oxygen administered, starting at 60 ft underwater.

sures of greater than two-and-a-half to three atmospheres so that this technique is limited to sixty feet.

Taking the case of an "infinite" stay (over four to five hours) in 300 feet of sea water, the ascent curve for the *seventy-five minute tissue* as calculated from Eq. (8) is shown in Fig. 3. At sixty feet we know the tissue is saturated at twice ($n \cong 2$) absolute 60-ft pressure or $2 \times (60+33) = 186$ ft. The diver may safely come out of the water when his tissues have a nitrogen saturation corresponding to twice atmospheric or 66 ft. Thus $P' = 188$, $P'' = 66$ and from Eq. (9), T'', the time spent in the Davis chamber, $= 113$ min. Comparing this modified ascent curve with the one calculated for air throughout (Fig. 3), it is apparent that this technique is very worth while for very long dives.

HELIUM-OXYGEN MIXTURES

In 1921 it was suggested that helium replace the nitrogen component in normal air for respiration at high pressures.[4] It was thought at this time that the lighter molecules would leave the tissues rapidly and cut down *decompression* time. Such is not necessarily the case[4] as a consideration of Eq. (8) will readily reveal.

In Eq. (8) the only factors that depend on the kind of gas used are n and $T_{\frac{1}{2}}$. Helium, with a diffusion rate five-halves that of nitrogen and a water solubility of only three-fifths, has a maximum $T_{\frac{1}{2}}$ of twenty to thirty minutes.[5] The normal duration of deep (200 ft or more) dives is generally not more than twenty to thirty minutes according to Navy regulations. For such short dives, the "slow" tissues do not play a major role in regulating the ascent so that helium does not give much assistance. Also, helium being more mobile appears to form bubbles more readily than nitrogen and requires a safe n value of 1.7 rather than 2.0. Taking as an example a twenty-minute stay at 300 ft, the tables[4] for helium give a total ascent time of one-hundred-and-thirteen minutes with pure oxygen breathing starting at the sixty-foot level. A normal air dive under identical conditions will take eighty-four minutes from the sea bottom to surface. Should the diver be trapped at a great depth for any length of time, however, his ascent would be much more rapid with helium than with air since his "slow" tissues would now determine the ascent curve. Helium is mainly advantageous in that it does not produce the narcotic effect under high pressures that is always associated with nitrogen-air.

DISCUSSION

Although this "classical" theory of *decompression* explains qualitatively a large body of phenomena associated with prevention of the *bends*, it is no longer trusted as a complete theory of nitrogen desaturation. There is doubt as to the physical reality of a variety of tissues having fixed half-saturation periods.[5] Actually it appears that when the body is suddenly moved from high to low pressure, nitrogen does not begin seeping from all tissues into the blood. Rather, the "slow" tissues rapidly accept nitrogen from the quickly desaturating tissues and thereby act as "buffers" against sudden bubble formation in these "fast" tissues. Wartime tests on high altitude *bends* indicate that a maximum $T_{\frac{1}{2}}$ of seventy-five minutes is dangerously conservative and that, in fact, this quantitity can vary in the same individual depending on his work and surrounding conditions.

While the n value of around 2 was fairly satisfactory for the conditions that Haldane considered (200-ft maximum depth), wartime research showed that n was variable for different tissues, being as high as 5 in "fast" tissues and as low as 1.7 in "slow" tissues. Also, n appears to be a function of total pressure as well, being around 1.75 at 200 feet underwater and 2.7 at high altitudes for the same tissue.

In spite of these added complexities, it must be stated that Haldane's contribution to the field was immensely important and showed a sure grasp of physical thought that was lacking in previous workers. His tables have saved many lives and permitted much useful work to be accomplished underwater and at high altitudes. Whether a completely accurate model of the physical processes occurring during bubble formation is possible remains to be seen, but the problem is an interesting one and its solution becomes more urgent as men push deeper under the sea.

[4] Navy Department, Bureau of Ships, *Diving Manual* (1944).

[5] J. F. Fulton, *Decompression Sickness* (W. B. Saunders Company, Philadelphia, 1951), Chap. III.

A Calculus of Measures and Its Application to the Relation between Rationalized and Unrationalized Electromagnetic Systems

J. R. Barker and R. O. Davies
Queen Mary College, London, E.1, England
(Received September 9, 1952)

When changing units and in dimensional analysis it is usual to employ the relation, physical quantity = measure × unit. Neither the "physical quantity" nor the "unit" is supposed to be a real number and yet the formalism is normally effective. This fact is given a rational explanation and the ideas involved are used to discuss certain questions connected with the rationalization of the electromagnetic equations.

INTRODUCTION

ALTHOUGH the Giorgi system of electrical units (now sponsored by the International Electrotechnical Commission) is gradually coming into widespread use, its relationship with the older unrationalized systems is still confused (see, e.g., Landolt and de Boer).[1] We believe that the trouble is caused by the use of the concept "physical quantity" which has objectionable metaphysical associations. Two variants of the rationalization process have been recognized, and it is necessary to refer to fundamental principles (on lines inspired by O'Rahilly)[2] in order fully to appreciate their relative merits.

PHYSICAL QUANTITIES, MEASURES, AND UNITS

There is a widespread notion (for which Maxwell was largely responsible) that "physical magnitudes" or "physical quantities" can be regarded as the product of "measures" (pure numbers) and "units" and that equations in physics are relations between "physical quantities." See, for example, Guggenheim[3] or Page.[4] Thus it has become fairly common to write

$$\begin{aligned} &\text{quantity} = \text{measure} \times \text{unit}, \\ &\{x\} = xx^{\dagger}, \end{aligned} \tag{1}$$

where neither $\{x\}$ nor $x^{\dagger}$ are supposed to be ordinary numbers. The "units" are closely connected with the "dimensions" of the corresponding quantity and are indeed separated out from equations of physical quantities and dealt with by methods similar to those of dimensional analysis. Such methods are normally foolproof and are, in fact, the quickest way of changing from one system of units to another. Unfortunately, the rational basis for these manipulations is rarely understood and this can lead to errors when discussing rationalization.

FUNDAMENTAL PRINCIPLES

The "measure" of a basic physical quantity (usually mass, length or time) is the ratio—found by a well-defined physical operation—between the quantity in question and an arbitrarily chosen conspecific unit. The measure of a derived quantity (perhaps already recognized in a qualitative sense) is found by means of an algebraic defining equation relating measures of basic quantities. This equation may also contain numerical constants—such as π—chosen for convenience. Measures are thus ordinary numbers, and here we shall denote them by plain letters.

We shall use the symbol $\{x\}$ for the verbal phrase "some particular example of the phenomenon whose measure is denoted by x," and the symbol $x^{\dagger}$ for the phrase "any example of the phenomenon whose measure is 1 on the chosen system of measurement," or, more shortly, "the unit of x." Different systems of measurement will be distinguished by subscripts.

For simplicity, the case of velocity, whose measure equation is $v = kl/t$, is considered. The "definitional constant" k is usually taken as 1, but its explicit inclusion here enables the arguments to be generalized without any difficulty. The measure of $\{v\}$ depends on the choice of k and of the units of length and time. For one particular choice we use the label 0 and regard

[1] M. Landolt and J. de Boer, Rev. Gén. de l'Électr. **60**, 499–508 (1951).
[2] A. O'Rahilly, *Electromagnetics* (Longmans Green and Company, London, 1938), first edition, Chap. XIV, XV.
[3] E. A. Guggenheim, Phil. Mag. **33**, 479–496 (1942).
[4] C. H. Page, Am. J. Phys. **20**, 1–4 (1952).

this system as a reference system. We use labels 1, 2, $\cdots$ for other choices and then for any of the systems labeled $0, 1, 2, \cdots, r, \cdots$, "measure-ratios" are defined as $[x_r]=x_0/x_r$, where x can be l, t, v, etc. In particular $[x_0]=1$. The following statements are then obvious:

(A) The measure equation $v_r=k_rl_r/t_r$ is valid (by definition) for each r.

(B) The measure-ratio equation $[v_r]=[k_r][l_r]/[t_r]$ is valid for each r.

(C) $$\frac{[l_r]}{[l_s]}=\frac{l_0/l_r}{l_0/l_s}=\frac{\text{measure of } l_r^\dagger \text{ in terms of } l_0^\dagger}{\text{measure of } l_s^\dagger \text{ in terms of } l_0^\dagger}$$

$$=\frac{l_s}{l_r}=\frac{\text{measure of } \{l\} \text{ in terms of } l_s^\dagger}{\text{measure of (the same) } \{l\} \text{ in terms of } l_r^\dagger};$$

$[l_r]/[l_s]$ is independent of the choice made for system 0.

(D) $$\frac{[v_r]}{[v_s]}=\frac{\text{measure of } v_r^\dagger \text{ in terms of } v_0^\dagger}{\text{measure of } v_s^\dagger \text{ in terms of } v_0^\dagger}.$$

(E) $v_0=v_1[v_1]=v_2[v_2]=\cdots=v_r[v_r]=\cdots$.

The same results hold for *any* derived quantity.

So far, we have only used the symbols $\{v\}$ and $v^\dagger$ as shorthand forms; we have not manipulated them in an algebraic manner. On searching among the purely algebraic numbers occurring in (A)–(E) above to find those whose manipulative properties are the same as those commonly ascribed to the "unit," the "measure" and the "physical quantity," the following conclusions can be drawn:

(a) The numbers $[l_1]$, $[l_2]$, $[t_1]$, $[t_2]$, etc., have properties formally identical with those of words like "foot" and "second" when these are manipulated in the traditional way. For example, if systems 1 and 2 have the foot and the yard as their respective units of length, (C) above is $3[l_1]=[l_2]$ (compare "3 feet = 1 yard"). It should be noticed that although $3[l_1]=[l_2]$ is an ordinary arithmetical equation, the numerical values of $[l_1]$ and $[l_2]$ depend on the choice of system 0; this fact is only important if two different reference systems are used (see later).

(b) The numbers l_1, v_1, etc., have their usual interpretation; that is, they are measures.

(c) Numbers such as v_0 have the same manipulative properties with respect to $[v_r]$ and v_r as "physical quantities" do with respect to "units" and "measures" on the orthodox treatment. For example, $v_0=v_1[v_1]$ corresponds to an equation like (1), $\{v\}=v_1v_1^\dagger$, which is not an ordinary algebraic equality. *Since the* 0-*system is entirely arbitrary, equations equivalent to those involving "physical quantities" have not the special significance that the words "physical quantity" appear to imply.*

The above discussion is illustrated by the solution of two problems:

(1) What is the measure on system 2 (yard, minute, $k=10$) of a velocity whose measure is 5 on system 1 (foot, second, $k=1$)?

In this case, $[l_2]=3[l_1]$, $[t_2]=60[t_1]$, and $[k_1]=10[k_2]$, so that

$$\begin{aligned}5[v_1]&=5[k_1][l_1]/[t_1]\\&=1000[k_2][l_2]/[t_2]\\&=1000[v_2].\end{aligned}$$

The answer is 1000 and the novel features are the use of purely arithmetical symbols and the manipulation of k.

(2) The power required to activate a solenoid of cross section A with current I through N turns of wire of density ρ, resistivity σ, and total winding mass M is

$$P=4\pi\rho\sigma A(NI)^2M^{-1}.$$

What is the power in watts when ρ is given in lb-ft^{-3}, σ in ohm cm, A in in.^{2}, I in amp, and M in lb? (Taken from Guggenheim.)[3]

Consider the following four numbered systems: (1) m, kg, sec, ohm; (2) in., kg, sec, ohm; (3) cm, kg, sec, ohm; (4) ft, lb, sec, ohm. The

problem is to find P_1 in terms of ρ_4, σ_3, A_2, I_1, and M_4. The arbitrary reference system is, as before, denoted by 0. Then,

$$\begin{aligned}P_1[P_1]=P_0&=4\pi\rho_0\sigma_0A_0(NI_0)^2M_0^{-1}\\&=4\pi\rho_4[\rho_4]\sigma_3[\sigma_3]A_2[A_2]\\&\qquad\times N^2I_1^2[I_1^2]^2M_4^{-1}[M_4]^{-1}\\&=4\pi\rho_4\sigma_3A_2(NI_1)^2M_4^{-1}\\&\qquad\times[\rho_4][\sigma_3][A_2][I_1]^2[M_4]^{-1}\\&=4\pi\rho_4\sigma_3A_2(NI_1)^2M_4^{-1}[M_4][L_4]^{-3}\\&\qquad\times[R_3][L_3][L_2]^2[I_1]^2[M_4]^{-1},\end{aligned}$$

where $[R]$ and $[L]$ are measure-ratios of resistance and length, respectively. But

$$[I_1]^2[R_3]=[I_1]^2[R_1]=[P_1],$$

so that

$$\begin{aligned}P_1[P_1]&=4\pi\rho_4\sigma_3A_2(NI_1)^2M_4^{-1}\\&\qquad\times[L_3][L_2]^2[L_4]^{-3}[P_1]\\&=\frac{4\pi}{30.48\times144}\times\frac{\rho_4\sigma_3A_2(NI_1)^2}{M_4}[P_1].\end{aligned}$$

The multiplier of $[P_1]$ in the last formula gives the measure in watts. In the conventional and simpler solution, given by Guggenheim, $\{P\}$ would be substituted for P_0 and "1 ohm" for $[R_3]$, etc.

These two examples are perhaps sufficient to show how the traditional method of dealing with changes of units can be justified. It is worth mentioning in passing that, by the use of measures and measure-ratios, the whole of dimensional analysis can be stated in terms of ordinary numbers. Such a treatment (as O'Rahilly[2] has shown) avoids many of the troublesome metaphysical questions associated with the orthodox procedure. Birkhoff[5] has also sketched a treatment of dimensional analysis in terms of ordinary numbers.

RATIONALIZATION

The form taken by the rationalized electromagnetic equations is already agreed upon, but some physicists treat the symbols as measures and others as "quantities." It is now necessary to examine the two variants of rationalization (discussed by Landolt and de Boer)[1] known as "rationalization of units" and "rationalization of quantities," to see the consequences of the two ways of interpreting the symbols. Only the case of the magnetic field $\{H\}$ is considered because the treatment of the other quantities affected by rationalization is fully analogous. Following the qualitative concept of field, the operation for its measurement is decided upon and a measure equation chosen so that numerical values can be assigned. One possible process defines the field in terms of the current $\{I\}$ flowing in a ring of radius $\{R\}$, the measure of $\{H\}$ at the center being given by the measure equation

$$H=AI/R, \tag{2}$$

which contains a definitional constant A. In what follows, current and length are taken as basic quantities. The measure H then depends on the choices made for A, for the unit of current, and for the unit of length.

Two choices of A are of interest: for the unrationalized system (denoted by u) $A=2\pi\equiv A_u$, for the rationalized system (denoted by r) $A=\frac{1}{2}\equiv A_r$; and also two different measuring systems for $\{I\}$ and $\{R\}$: for the e.m. system (denoted by m), $\{I\}$ measured in emu, $\{R\}$ in cm, for the practical system (denoted by p), $\{I\}$ measured in amp, $\{R\}$ in meters. With these choices, there are four possible measures for the magnetic field:

$$\begin{aligned}H_{um}&=\frac{A_uI_m}{R_m},\quad H_{up}=\frac{A_uI_p}{R_p},\\H_{rm}&=\frac{A_rI_m}{R_m},\quad H_{rp}=\frac{A_rI_p}{R_p}.\end{aligned} \tag{3}$$

To convert from one measure of $\{H\}$ to another we already have $A_u/A_r=4\pi$ and need in addition the measure-ratios $I_p/I_m=10$ and $R_m/R_p=100$. (The latter ratios are the inverses of the ratios of the measures of the units—the measures being expressed in terms of an arbitrary conspecific unit in accordance with (C).) Either directly from Eqs. (3) or else by using the measure-ratio equation $[H]=[A][I]/[R]$, it is found that

$$H_{um}=\frac{1}{1000}H_{up}=4\pi H_{rm}=\frac{4\pi}{1000}H_{rp}. \tag{4}$$

[5] G. Birkhoff, *Hydrodynamics* (Princeton University Press, Princeton, New Jersey, 1950), first edition, Chap. III.

FIRST VARIANT—RATIONALIZATION OF UNITS

A reference system 0 is chosen in which the measure equation

$$H_0 = A_0 I_0 / R_0, \tag{5}$$

corresponds to an "equation of quantities":

$$A_u = 2\pi, \quad A_r = \tfrac{1}{2},$$

so that

$$[A_u] = A_0/2\pi \quad \text{and} \quad [A_r] = 2A_0;$$

also

$$[H_u] = [A_u][I]/[R], \quad [H_r] = [A_r][I]/[R]. \tag{6}$$

Hence

$$[H_u] = \frac{1}{4\pi}[H_r], \tag{7}$$

where the labels m or p are omitted since either could be used.

Also

$$[H_m] = [A][I_m]/[R_m]$$

and

$$[H_p] = [A][I_p]/[R_p],$$

but

$$[I_m]/[I_p] = 10 \quad \text{and} \quad [R_p]/[R_m] = 100.$$

Hence

$$[H_m] = 1000[H_p], \tag{8}$$

where the labels u and r are omitted since either could be used. From Eqs. (7) and (8) it follows that

$$[H_{um}] = 1000[H_{up}] = \frac{1}{4\pi}[H_{rm}] = \frac{1000}{4\pi}[H_{rp}], \tag{9}$$

which correspond to "unit" equations consistent with the measure equations (4). Conventional methods can be used to recover Eqs. (9) on condition that the following device is adopted. Introduce the words "rat" and "unrat" related to $[A_r]$ and $[A_u]$, respectively, in the same way as the word "yard" is related to the number $[l_2]$ in example 1 above. Just as "3 feet = 1 yard" so "1 rat = 4π unrat." Then Eqs. (9) correspond to

$$1 \text{ oersted}^* = 1 \text{ unrat } \frac{\text{emu}}{\text{cm}}$$

$$= 1 \text{ unrat } \frac{10 \text{ amp}}{\frac{1}{100} \text{ m}} = 1000 \text{ unrat } \frac{\text{amp}^*}{\text{m}}$$

$$= \frac{1}{4\pi} \text{ rat } \frac{\text{emu}}{\text{cm}} = \frac{1000}{4\pi} \text{ rat } \frac{\text{amp}^*}{\text{m}}, \tag{10}$$

where the terms marked * correspond, in order, to those in Eqs. (9).

Noting that

$$[H_u] = \frac{A_0}{2\pi}\frac{[I]}{[R]} \quad \text{and} \quad [H_r] = 2A_0\frac{[I]}{[R]},$$

where the value of A_0 is arbitrary, other permissible sets of names for the four units of $\{H\}$ (some of which have been used in the literature) can be given. Thus

$$\left\{\begin{array}{llll} H_{um}^{\dagger}: & \frac{1}{4\pi} \text{ emu/cm}, & H_{rm}^{\dagger}: & 1 \text{ emu/cm} \\ H_{up}^{\dagger}: & \frac{1}{4\pi} \text{ amp/m} & H_{rp}^{\dagger}: & 1 \text{ amp/m} \end{array}\right\}$$

is one possibility,

and

$$\left\{\begin{array}{llll} H_{um}^{\dagger}: & 1 \text{ emu/cm}, & H_{rm}^{\dagger}: & 4\pi \text{ emu/cm} \\ H_{up}^{\dagger}: & 1 \text{ amp/m}, & H_{rp}^{\dagger}: & 4\pi \text{ amp/m} \end{array}\right\}$$

is another.

It is not permissible to take the names from *different* sets (e.g., $H_{rp}^{\dagger}$: 1 amp/m and $H_{um}^{\dagger}$: 1 emu/cm) and then to expect the arithmetical properties of the names to be preserved. This was indeed the difficulty encountered by Landolt and de Boer.[1] (We suggest that the *best* way of naming the units is to adopt the "rat" and "unrat" as indicated above, since this preserves the arbitrary nature of A_0.)

As there is only one reference system 0, there is only one "equation of quantities" corresponding to Eq. (5). (The rationalization process has been one of changing units and measures in in-

verse ratios.) By suitable choice of the arbitrary A_0, the "equation of quantities" can be given the same form as the rationalized or unrationalized equation of measures but not both simultaneously. The process of rationalization of units is entirely logical and satisfactory to those who use measures exclusively, but it is extremely inconvenient for those who wish to interpret both

$$H=2\pi I/R \quad \text{and} \quad H=\tfrac{1}{2}I/R$$

as equations of quantities. The desire to use the idea of "quantity" has apparently led to the invention of the second variant of the rationalization process, which will now be discussed.

SECOND VARIANT—RATIONALIZATION OF QUANTITIES

In this variant two measures of $\{H\}$ are defined by equations in which the definitional constants are chosen and kept fixed. What we call the "unrationalized" and "rationalized measures" of $\{H\}$ are defined by

$$H=2\pi I/R \quad \text{(unrationalized)} \tag{11}$$

and

$$H'=\tfrac{1}{2}I'/R' \quad \text{(rationalized).} \tag{12}$$

To each of these there corresponds a separate reference system in which the measures are

$$H_0=2\pi I_0/R_0 \quad \text{(system 0, unrationalized),} \tag{13}$$

$$H_0'=\tfrac{1}{2}I_0'/R_0' \quad \text{(system 0', rationalized).} \tag{14}$$

These analogs of equations of quantities have the same form as measure equations (11) and (12), and the quantities themselves have been called the "unrationalized" and "rationalized magnetic field."

Changes of units (and measures) between *either* unrationalized systems *or* rationalized ones are carried out in the normal way, e.g., $[H_m]=[I_m]/[R_m]=1000[I_p]/[R_p]=1000[H_p]$ (unrationalized case), and similarly

$$[H_m']'=1000[H_p']' \quad \text{(rationalized case).}$$

Because $[H_m]=[I_m]/[R_m]$ and $[H_m']'=[I_m']/[R_m']$, these measure-ratios for $\{H\}$ can be given the same name, emu/cm. Similarly the two practical units can both be called the amp/m. Clearly the unit calculus cannot be used to transform unrationalized measures to rationalized ones or vice versa.

In order to transform between the four chief measures of $\{H\}$ (to recover Eqs. (4)) the following must be noted:

(a) The unrationalized e.m. and practical measures of $\{H\}$ are related inversely as their units, and similarly for the rationalized measures.

(b) Unrationalized and rationalized measures of $\{H\}$, if $\{I\}$ and $\{R\}$ are measured in the same units in both cases, are related by

$$H=4\pi H'. \tag{15a}$$

(c) Users of quantities (instead of rationalizing measures as in (b)) will rationalize quantities using the equation of quantities analogous to

$$H_0=4\pi H_0', \tag{15b}$$

but the result is the same as in (b).

RATIONALIZATION—SUMMARY

Rationalization of units	*Rationalization of quantities*
The magnetic field has four measures and four unit names:	The field has two unrationalized measures and two unit names:
H_{um} unrat emu/cm (oersted)	H_m emu/cm (oersted)
H_{up} unrat amp/m	H_p amp/m
	The field has two rationalized measures and two unit names:
H_{rm} rat emu/cm	H_m' emu/cm
H_{rp} rat amp/m	H_p' amp/m
Also, 1 rat $=4\pi$ unrat	Also, $H_m=4\pi H_m'$ and $H_p=4\pi H_p'$

Common to both variants are
1 emu = 10 amp, 1 m = 100 cm.

Therefore

$$H_{um}(=H_m)=\frac{1}{1000}H_{up}\left(=\frac{1}{1000}H_p\right)$$

$$=4\pi H_{rm}(=4\pi H_m')=\frac{4\pi}{1000}H_{rp}\left(=\frac{4\pi}{1000}H_p'\right)$$

in agreement with Eqs. (4).

As an example, the ways of specifying the field at the center of a single circular turn of radius 1 cm carrying a current of 1 amp are given:

(a) rationalization of units,

$$H=6.28\times10^{-1}\text{ unrat emu/cm}$$
$$=6.28\times10^{2}\text{ unrat amp/m}$$
$$=5\times10^{-2}\text{ rat emu/cm}=5\times10\text{ rat amp/m.}$$

(b) rationalization of quantities,

magnetic field (unrationalized)
$$=6.28\times10^{-1}\text{ emu/cm}=6.28\times10^{2}\text{ amp/m,}$$

magnetic field (rationalized)
$$=5\times10^{-2}\text{ emu/cm}=5\times10\text{ amp/m.}$$

The same numbers (measures) appear whichever variant is used.

CONCLUSIONS

A calculus of measures has been presented which is complete in itself. It has been applied to explain the successes of (a) the interpretation of the symbols in equations as "physical quantities" or "magnitudes" and (b) the manipulation of unit names when changing units. It is quite unnecessary to use "quantities" and, in view of the metaphysical associations of the word, dangerous when investigating unsettled ground. The manipulation of unit names does, however, provide the easiest way of changing units and can be regarded as a mnemonic for the more logical manipulation of ordinary numbers (i.e., the measure-ratios) in accordance with the rules of ordinary algebra.

We have presented two variants of the rationalization process. Those who insist on using "equations of quantities" must employ the second variant because only this one provides *two* equations of quantities (13) and (14) of exactly the same form as the measure equations (11) and (12), respectively. Those who use measures exclusively can use either variant with success. In practice the choice is immaterial. It must, however, be clearly stated either in the name of the unit (first variant) or in the name of the measure or quantity (second variant) whether or not rationalization has been employed.

To Secretaries of Regional Sections of AAPT

The editor is very anxious to publish reports of Section meetings as promptly as possible after they are held. There is an unavoidable delay of about three months from the time a manuscript leaves my office till the time of its appearance in print. These three months are consumed by work in the Publications Office of the Institute, in the printing house, and by the reading of proof, and there is no simple way of shortening the period.

You can, however, help to assure the most rapid publication of your Section news by sending me a report of each meeting immediately after the close of the meeting. To save time, the report should be written following meticulously the standard pattern that we use, as for example, on pages 73, 74 of the January, 1953 issue or pages 461–463 of the October, 1952 issue. The material must be typewritten and everything must be double-spaced to allow for editorial markings. Please note that this means *everything*. The contributed papers need not be numbered serially if there are only a few. Numbering is helpful if there are more than six or eight.

Abstracts of papers should be included whenever they are available and I would urge secretaries of regional Sections to collect such abstracts either before or at the actual meeting. Abstracts which deal with abstruse research problems may well be omitted in order that those which deal with matters of teaching may be made a little longer. Any work that you do in shaping, clarifying, and improving abstracts will be much appreciated.

THOMAS H. OSGOOD
Editor

Longitudinal Vibrations of a Vertical Column by the Method of Laplace Transform

L. H. Hall
Santa Barbara College, University of California, Santa Barbara, California
(Received October 15, 1952)

An interesting illustrative problem in mechanics is the motion executed by a vertical rod or column when it suffers sudden loss of support at the bottom. A solution is presented by the method of Laplace transform in two alternative forms: one represents the motion as a superposition of free fall and two oppositely directed progressive waves equal in amplitude and form; the other, as a free fall and a pulse, initiated at the bottom and propagated up and down the column upon reflection at the ends.

PHYSICALLY a simple type of problem, whose solution may deserve illustration in a course in mechanics or applied mathematics, is the motion executed by a vertical, elastic column when it suffers sudden loss of support. Since the Laplace transformation applied to this case leads to a solution intelligible and natural from a physical point of view, the problem serves well also to illustrate this important method which today increasingly appears in the literature of physics and engineering. (In a recent account of transform applications,[1] for instance, are presented examples in atomic and nuclear physics, hydrodynamics, elasticity, vibrations, and heat conduction.)

DERIVATION OF THE EQUATION OF MOTION

Consider a vertical, elastic rod or column of length L cross section A. Let it be subject to atmospheric pressure P_0; initially supported at the bottom, and in static equilibrium under gravity forces. Suppose the support to be removed suddenly. This action will be represented as an instantaneous drop in pressure at the bottom of the column from the static value, $P_0+\rho gx$, to the atmospheric value P_0. We take the case of negligible internal and external friction. Let the top of the column be initially at the origin; x = the distance from the origin to any point or particle on the column under consideration; u = particle displacement; v = total particle velocity; P = total pressure; $P_e=P-P_0$, the excess pressure; ρ = mass density; w = weight density; and E = Young's modulus. With reference to Fig. 1, a small slice of medium lying between x and $x+\Delta x$ experiences a net force, $(-\partial P/\partial x)\Delta xA+w\Delta xA$. By Newton's law we have

$$(\Delta xA\rho)\frac{d^2u}{dt^2}=-\frac{\partial P}{\partial x}\Delta xA+w\Delta xA,$$
$$\rho\frac{d^2u}{dt^2}=-\frac{\partial P}{\partial x}+w. \tag{1}$$

Since the pressure at any point is P_0+P_e, $\partial P/\partial x=\partial P_e/\partial x$. The fractional change in volume of the slice, $\Delta V/V$, is simply $\partial u/\partial x$. Taking the usual linear relation between the excess pressure P_e and the relative compression, we have $P_e=-E\Delta V/V=-E\partial u/\partial x$, and therefore $\partial P/\partial x=-E\partial^2u/\partial x^2$. Letting $a^2=E/\rho$ and limiting study to cases where the motion is small enough so that total time derivatives can be approximated by partial time derivatives, which is true in the usual applications, we arrive at the equation of motion,

$$\frac{\partial^2u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2u}{\partial t^2}=-\frac{g}{a^2}. \tag{2}$$

The initial and boundary conditions are as follows: Initially the column is at rest, so the total velocity $v=\partial u/\partial t=0$. The pressure has the simple static distribution, P_0+wx or $P_e=wx$; associated with this excess pressure there is an elastic dis-

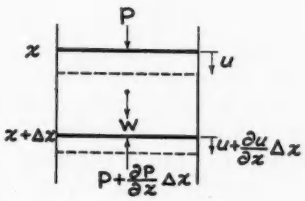


Fig. 1. Volume element of medium under compression.

[1] I. N. Sneddon, *Fourier Transforms* (McGraw-Hill Book Company, Inc., New York, 1951), first edition.

placement of the fluid particles u. It is convenient to let $u=0$ when $x=0$; then, for compression, u will be negative when x is positive. The initial displacement is $-x^2g/2a^2$, which can be verified by noting that it satisfies the stress-strain relation, $P_e=-E\partial u/\partial x$, and gives the required value zero, when x is zero. When the pressure at the bottom is instantaneously dropped to P_0, we have $P_e=0$, $\partial u/\partial x=0$, when $x=L$ and when $x=0$. We have therefore the system of equations

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}=\frac{-g}{a^2};\\ &\frac{\partial u}{\partial t}=0;\quad u=\frac{-x^2g}{2a^2}\quad\text{when}\quad t=0;\\ &P_e=0;\quad \frac{\partial u}{\partial x}=0,\quad\text{at } x \text{ equal to 0 or to } L\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{when } t>0.\end{aligned}\right\}\quad(3)$$

THE METHOD OF LAPLACE TRANSFORMATION

It may not be amiss to review the idea and some of the definitions of transform methods. Sneddon[1] gives a nice orientation in his opening chapter, and we follow his train of thought. Given a set of equations which define implicitly a desired function, it is a common technique to introduce a transformation of some kind to facilitate the solution. An integral transformation defined as follows is one of the useful types: Let $f(t)$ be a function of the real variable t, which ranges over the interval $(0, \infty)$, and let $K(p, t)$ be a selected function of t and the variable p, which may be complex. When the integral

$$\int_0^\infty f(t)K(p, t)dt$$

is convergent, it defines a function $\bar{f}(p)$ of the variable p, which is termed the integral transform of the function $f(t)$ by the kernal $K(p, t)$. Simple and serviceable kernals are e^{-pt}, t^{p-1}, $\cos(pt)$, and $\sin(pt)$. The first of these is the Laplace kernal giving the Laplace transformation; the second, the Mellin kernal; the others, Fourier cos and sin kernals. In a problem we seek to transform the determining equation for the desired $f(t)$ into an equation which the transform function, $\bar{f}(p)$, must satisfy. It is readily verified that a differentiation of the original function, $f(t)$, corresponds merely to a multiplication of the Laplace transform by its independent variable p with the addition of a constant, which for a very broad class of functions is $-f(0)$. (Further, in many applications we can adjust conditions so that $f(0)=0$.) Upon solving the transform equation, we need of course a way back from $\bar{f}(p)$ to $f(t)$. The more elementary and frequent transform pairs are cataloged in transform tables; for other cases we look to inversion theorems such as the Fourier-Mellin inversion theorem, which will be employed later. For linear differential equations with constant coefficients, a simple and customary way to effect the transition from the equation for the original function to that for the transform or image function, is to multiply through the original equation by e^{-pt} and to integrate with respect to t from 0 to ∞. In the Eqs. (3) u is a function of two independent variables rather than one; we shall transform with respect to only one, t. Multiplying Eqs. (3) by e^{-pt} and then integrating from 0 to ∞:

$$\int_0^\infty e^{-pt}\frac{\partial^2 u}{\partial x^2}dt-\frac{1}{a^2}\int_0^\infty e^{-pt}\frac{\partial^2 u}{\partial t^2}dt=\frac{-g}{a^2}\int_0^\infty e^{-pt}dt.$$

For the first term,

$$\int_0^\infty e^{-pt}\frac{\partial^2 u}{\partial x^2}dt=\frac{\partial^2}{\partial x^2}\int_0^\infty e^{-pt}udt=\frac{\partial^2\bar{u}}{\partial x^2}$$

(by definition).

For the second term, integrating by parts,

$$\frac{1}{a^2}\int_0^\infty e^{-pt}\frac{\partial^2 u}{\partial t^2}dt=\frac{1}{a^2}\left[e^{-pt}\frac{\partial u}{\partial t}\right]_0^\infty+\frac{p}{a^2}[e^{-pt}u]_0^\infty+\frac{p^2}{a^2}\int_0^\infty e^{-pt}udt.$$

The first number is zero, for $\partial u/\partial t=0$, when $t=0$ and $(e^{-pt}\partial u/\partial t)\to 0$ as $t\to\infty$, because the factor e^{-pt} becomes small more rapidly then $\partial u/\partial t$ becomes large with increasing t. The second member yields $pgx^2/2a^4$. The third member is $p^2\bar{u}/a^2$. Finally,

$$\frac{-g}{a^2}\int_0^\infty e^{-pt}dt=\frac{-g}{a^2}\frac{1}{p}.$$

Hence the equation to be satisfied by the Laplace transform (the so-called subsidiary equation) is an ordinary differential equation in x,

$$\frac{\partial^2\bar{u}}{\partial x^2}-\frac{p^2}{a^2}\bar{u}=\frac{px^2g}{2a^4}-\frac{g}{a^2p} \tag{4}$$

with boundary conditions $\partial\bar{u}/\partial x=0$, when $x=0$ or L. These boundary conditions follow from $\partial u/\partial x=0$ at x equal to 0 or L, since

$$\frac{\partial\bar{u}}{\partial x}=\frac{\partial}{\partial x}\int_0^\infty e^{-pt}u\,dt=\int_0^\infty e^{-pt}\frac{\partial u}{\partial x}dt=0.$$

The homogeneous part of Eq. (4) has the familiar solution

$$\bar{u}_{\text{hom}}=Ae^{px/a}+Be^{-px/a}=A\cosh\left(\frac{px}{a}-\theta\right).$$

By inspection, a particular solution is $\bar{u}_{\text{part.}}=-gx^2/2a^2p$. Thus the solution is

$$\bar{u}=A\cosh\left(\frac{px}{a}-\theta\right)-\frac{gx^2}{2a^2p}. \tag{5}$$

Introducing the boundary conditions, $\partial\bar{u}/\partial x=0$ when $x=0$ or L:

$$\frac{\partial\bar{u}}{\partial x}=\frac{p}{a}A\sinh\left(\frac{px}{a}-\theta\right)-\frac{xg}{a^2p}.$$

When $x=0$,

$$\frac{\partial\bar{u}}{\partial x}=\frac{p}{a}A\sinh(-\theta)=0;\quad\text{therefore}\quad\theta=0.$$

When $x=L$,

$$\frac{\partial u}{\partial x}=\frac{p}{a}A\sinh(pL/a)-\frac{Lg}{a^2p}=0;$$

therefore

$$A=\frac{Lg}{ap^2\sinh(pL/a)}.$$

Thus the solution of the subsidiary Eq. (4) with boundary conditions is

$$\bar{u}=\frac{Lg}{ap^2}\frac{\cosh(px/a)}{\sinh(pL/a)}-\frac{x^2g}{2a^2p}. \tag{6}$$

Then u, the inverse Laplace transform of $\bar{u}$, is equal to the sum of the inverse transforms of the two right-hand terms of Eq. (6). It is readily verified that the inverse transform of $1/p$ is 1, and thus that the inverse of the second term is $-x^2g/2a^2$. For the inverse transform of the more complicated first term, unlikely to be found in a table, we turn to the Fourier-Mellin inversion theorem,[2–4] which states that

$$u(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{pt}\bar{u}(p)dp,$$

where γ is any constant greater than the real parts of all the singularities of $\bar{u}(p)$; i.e., integration is along any line parallel to the imaginary axis and to the right of the singularities. Thus we have to evaluate the complex integral

$$\frac{Lg}{2\pi ia}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{pt}\frac{\cosh(px/a)}{\sinh(pL/a)}\frac{dp}{p^2}. \tag{7}$$

This will be done in two ways: (1) by series integration; (2) by residue integration.

EXPONENTIAL SERIES EXPANSION OR HEAVISIDE WAVE FORM

The physical action at the removal of support (an abrupt drop in pressure) suggests the idea of a pressure pulse (subpressure) which is initiated at the end where $x=L$, travels toward the end where $x=0$, is reflected, travels back and is reflected again at the bottom, etc. If the integral (7) is evaluated by expressing the integrand in a series of exponentials (the so-called Heaviside wave method) we are led to a solution whose form expresses these physical features.

The integrand of the inversion integral (7)

[2] R. V. Churchill, *Modern Operational Mathematics in Engineering* (McGraw-Hill Book Company, Inc., New York, 1944), first edition.

[3] H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics* (Oxford University Press, London, 1941), first edition.

[4] M. F. Gardner and J. L. Barnes, *Transients in Linear Systems* (John Wiley and Sons, Inc., New York, 1942), first edtion, Vol. 1.

may be expanded as follows:

$$e^{pt}\frac{\cosh(px/a)}{\sinh(pL/a)}=e^{pt}\frac{(e^{px/a}+e^{-px/a})}{(e^{pL/a}-e^{-pL/a})}$$
$$=e^{pt}\frac{e^{-pL/a}(e^{px/a}+e^{-px/a})}{(1-e^{-2pL/a})}$$
$$=e^{pt-pL/a}(e^{px/a}+e^{-px/a})$$
$$\times(1+e^{-2pL/a}+e^{-4pL/a}+\cdots)$$
$$=\sum_{m=0}^{\infty} e^{p(at-(2m+1)L+x)/a}+e^{p(at-(2m+1)L-x)/a}.$$

The displacement may, therefore, be written

$$u=\frac{Lg}{2\pi ia}\int_{\gamma-i\infty}^{\gamma+i\infty}\sum_{m=0}^{\infty}\{e^{p(at-(2m+1)L+x)/a}+e^{p(at-(2m+1)L-x)/a}\}\frac{dp}{p^2}-\frac{x^2g}{2a^2}.$$

The excess pressure

$$\left[=-E\frac{\partial u}{\partial x}=-\frac{wa^2}{g}\frac{\partial u}{\partial x}\right]$$

is then

$$P_e=wx-\frac{wL}{2\pi i}\int_{-i\infty}^{\gamma+i\infty}\sum_{m=0}^{\infty}\{e^{p(at-(2m+1)L+x)/a}-e^{p(at-(2m+1)L-x)/a}\}\frac{dp}{p}. \quad (8)$$

The integral consists of a summation of terms of the form

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{kp}\frac{dp}{p},$$

which has the value 1, if $k>0$ and the value 0, if $k<0$ (refeference 3, p. 165), i.e., the unit step function, $U(k)$. Replacing k, we have $U(t-[(2m+1)L\pm x]/a)$, which equals 0 when $t<[(2m+1)L\pm x]/a$ and equals 1 when $t>[(2m+1)L\pm x]/a$. The excess pressure distribution accordingly may be written

$$P_e=xw-wL\sum_{m=0}^{\infty}U\left(t-\frac{(2m+1)L-x}{a}\right)-U\left(t-\frac{(2m+1)L+x}{a}\right). \quad (9)$$

The total pressure is simply P_0+P_e.

The fluid velocity distribution, $v[=v_e=\partial u/\partial t]$, is

$$v=\frac{Lg}{a}\sum_{m=0}^{\infty}U\left(t-\frac{(2m+1)L-x}{a}\right)+U\left(t-\frac{(2m+1)L+x}{a}\right). \quad (10)$$

DISCUSSION OF HEAVISIDE WAVE SOLUTION

The displacement distribution will be discussed in the next section.

Pressure Distribution

To interpret expression (9), consider some point along the column, say the mid-point, $x=L/2$, and let the time taken for a disturbance to traverse the length of the column be $\tau=L/a$; then the expression (9) becomes

$$P_e=\frac{wL}{2}-wL\sum_{m=0}^{\infty}U(t-(2m\tau+\tau/2))-U(t-(2m\tau+3\tau/2)). \quad (11)$$

Consider now a time lapse less than that required for an elastic disturbance to travel from the end $x=L$ to the mid-point of the column. This is a time for which $t<\tau/2$, and hence for which $t<(2m\tau+\tau/2)$ for all values of m. Thus Eq. (11) reduces to $P_e=wL/2$, the correct initial pressure at the mid-point. When $t>\tau/2$ but $t<3\tau/2$, $P_e=wL/2-wL(1-0)=-wL/2$. When $t>3\tau/2$ but $t<5\tau/2$, $P_e=wL/2-wL(1-1)=wL/2$. When $t>5\tau/2$ but $t<7\tau/2$, $P_e=wL/2-wL(1+1-1)=-wL/2$. The interpretation is clearly that a pulse of subpressure wL is initiated at the bottom at the withdrawal of support, $t=0$; and that when a lapse of time sufficient for the pulse to reach point $x=L/2$ has occurred ($t>\tau/2$), this subpressure arrives and is superimposed on the original pressure of $wL/2$, leaving a net pressure of $-wL/2$. When a time lapse such that $t>3\tau/2$ but $<5\tau/2$ has occurred, the pulse has been reflected from end $x=0$; has returned as a superpressure, restoring the initial value of the pressure at $x=L/2$. When $t>5\tau/2$ but $<7\tau/2$, the pulse has passed on to the end $x=L$; has been reflected as a subpressure again; and has again returned to the mid-point causing the

pressure to drop to $-wL/2$. In this manner the pressure pulse travels up and down the column, causing on upward passage a pressure decrement $-wL$, which added to the existing pressure $wL/2$, gives a resultant pressure $-wL/2$; and causing on its downward journey a pressure increment wL, restoring the original positive pressure $wL/2$. The pulse of superpressure is reflected at the ends as a pulse of subpressure and vice versa. Near the bottom of the column, $x=L-\Delta L$, the excess pressure fluctuates discontinuously between 0 and $wL-\Delta P$. Pressure history diagrams for three points along the rod are shown in Fig. 2.

Velocity Distribution

When $x=L/2$, Eq. (10) yields the information: the velocity jumps from 0 to $g\tau$ for $t>\tau/2$ and remains at this value until $t>3\tau/2$ when it jumps to $2g\tau$. It continues to increase in increments of $g\tau$ every time interval τ, which obviously corresponds to the arrival of a velocity pulse upon reflection at the ends. When $x=L$ the velocity jumps from 0 to $g\tau$ for $t>0$ and thereafter increases in increments of $g\tau$ every time interval 2τ.

INVERSION BY RESIDUE INTEGRATION

The singularities of the integrand in Eq. (7) are at the zeros of $\sinh(pL/a)$ which give simple poles at $(pL/a)=\pm n\pi i$, n integral, and at $p=0$. The residues for the former singularities are given by the formula[5]

$$\sum_{n=-\infty}^{n=+\infty} \frac{Lg}{2\pi i a}\left[\frac{e^{pt}\cosh(px/a)}{\dfrac{d}{dp}p^2\sinh(Lp/a)}\right]_{(Lp/a=\pm n\pi i)}.$$

With the help of the relations between hyperbolic, circular, and exponential functions this reduces to

$$\frac{1}{2\pi i}\cdot\frac{2L^2g}{\pi^2a^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\times\cos(n\pi at/L)\cos(n\pi x/L). \quad (12)$$

[5] R. V. Churchill, *Introduction to Complex Variables and Applications* (McGraw-Hill Book Company, Inc., New York, 1948), first edition, p. 122.

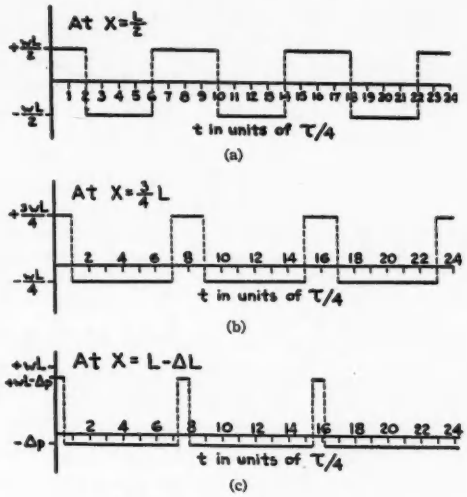


FIG. 2. Pressure history curves.

For the residue at $p=0$, we form the Laurent expansion of the integrand about $p=0$ by expanding the factors individually and combining,

$$\frac{Lg}{2\pi i a}\cdot\frac{e^{pt}\cosh(Lx/a)}{p^2\sinh(Lp/a)}=\frac{Lg}{2\pi i a}\frac{(1+p+p^2t^2/2+\cdots)}{p^2}\times(1+p^2x^2/2a^2+\cdots)(a/Lp-Lp/6a+\cdots),$$

which gives for the coefficient of the $1/p$ term

$$\frac{1}{2\pi i}\left[-\frac{L^2g}{6a^2}+\frac{x^2g}{2a^2}+\frac{t^2g}{2}\right]. \quad (13)$$

The integral (7) equals the sum of the residues of the integrand, expressions (12) and (13), multiplied by $2\pi i$. We get finally

$$u=-\frac{L^2g}{6a^2}+\frac{2L^2g}{\pi^2a^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\times\cos(n\pi at/L)\cos(n\pi x/L)+\tfrac{1}{2}gt^2. \quad (14)$$

Applying a trigonometric identity, the solution for the displacement becomes

$$u=-\frac{L^2g}{6a^2}+\frac{L^2g}{\pi^2a^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\times\left\{\cos\frac{n\pi}{L}(at-x)+\cos\frac{n\pi}{L}(at+x)\right\}+\tfrac{1}{2}gt^2. \quad (15)$$

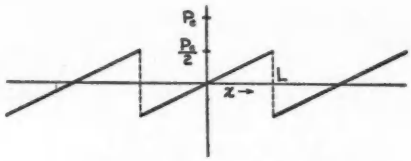


FIG. 3. Pressure wave form when $t=0$.

The excess pressure $[=-(wa^2/g)(\partial u/\partial x)]$ is

$$P_e=\frac{wL}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\times\left\{\sin\frac{n\pi}{L}(at-x)-\sin\frac{n\pi}{L}(at+x)\right\}. \quad (16)$$

The fluid velocity is

$$v=\frac{Lg}{\pi a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\times\left\{\sin\frac{n\pi}{L}(at-x)+\sin\frac{n\pi}{L}(at+x)\right\}+gt. \quad (17)$$

DISCUSSION OF SOLUTIONS

Displacement Distribution

When $t=0$, Eq. (14) or (15) reduces to

$$u=-\frac{L^2g}{6a^2}+\frac{2L^2g}{\pi^2a^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x/L),$$

which is just the Fourier expansion of the function $u=-gx^2/2a^2$, the initial displacement distribution along the column.[6] For each n, the trigonometric summation term in Eq. (14) gives a harmonic standing wave motion; and for all n, a set of superimposed standing waves, constituting, a complex standing wave or vibration of the column. The last term is the familiar free fall motion. Thus the column vibrates as it falls.

Pressure Distribution

When $t=0$, Eq. (16) is the Fourier series representation of wx, the initial pressure distribution along the column. Terms of the form, $\sin n\pi(at-x)/L$, represent progressive harmonic waves traveling in the positive direction, and the sum, $\sum \sin n\pi(at-x)/L$, a series of harmonic waves—the harmonic components of a wave of more complex form. When $t=0$, this wave is pictured in Fig. 3. As time passes the wave form moves to the right with velocity a. Similarly $\sum \sin n\pi(at+x)$ represents a wave of the same form but moving in the negative direction, left. The two waves coincide in initial position when $t=0$ and together give $\frac{1}{2}P_e+\frac{1}{2}P_e=P_e$. They are illustrated in Fig. 3. The ensuing distribution of pressure for any time t can be obtained graphically by sliding one of the superimposed wave forms to the right a distance x equal to at, and the other an equal distance to the left and adding ordinates in the new positions to give the new pressure distribution. This recipe gives pressure history curves identical, of course, with those derived from the Heaviside solution.

While the problem at hand is as readily solved by the method of Fourier series as by the Laplace transformation, there are in many cases advantages to the latter. In a general way this may be seen in the fact that differentiation of a function has merely the effect of multiplying the associated Laplace transform by its independent variable and adding to it the constant, $-f(0)$. Thus, in the case of linear differential equations with constant coefficients, the transformation converts ordinary differential equations into algebraic equations and partial differential equations in two independent variables into ordinary differential equations. In many instances simplification results also for linear differential equations with variable coefficients and for integral equations. Nonlinear differential equations, however, are less likely to be good subjects. Since the integral of a discontinuous function may be a continuous or analytic one, the Laplace transformation also enables the study of discontinuous functions (such as occur in impulse or transient phenomena) through the medium of the more sedate transform functions. To be sure there does arise the added problem of evaluating the inversion integral when the transform is not one already cataloged. (It was this inversion task which may have made the solution of the present problem seem cumbersome.) However, the exchange of a direct integration for an appreciable simplification of a differential equation is frequently a good mathematical bargain.

[6] W. E. Byerly, *Fourier Series and Spherical Harmonics* (Ginn and Company, New York, 1893), p. 51.

A Proposed Procedure for Selecting and Using Symbols for Physical Units

DUANE ROLLER
Hughes Research and Development Laboratories, Culver City, California
(Received November 24, 1952)

Since the symbols—names, or their abbreviations—for physical units may be incorporated in physical equations in exactly the same manner as are the letter symbols for physical quantities and the numerical symbols, it follows that the rules for selecting and using these unit-symbols should be formulated in the light of established mathematical procedures. A set of such rules is proposed. Symbols are suggested for 185 commonly used units and combining forms.

VARIOUS standardizing and publication agencies in the physical sciences and engineering have recommended or adopted lists of "abbreviations" for the names of physical units. None of the lists, even the most recent ones, seem to have been compiled with much regard for the view that these "abbreviations" may be usefully employed as *physical symbols*, in the sense that they may be incorporated in physical equations in exactly the same manner as are the letter symbols for physical quantities and the numerical symbols; moreover, that these symbols (abbreviations) are then subject to the same conventions and rules of operation as are the other mathematical symbols of physics. For this view to hold, a letter symbol for a physical quantity must be interpreted as representing both the magnitude and unit of the quantity. It is true that the letter symbols may be regarded alternatively as representing magnitudes alone. Either one of the interpretations may be adopted, so long as one uses it exclusively throughout a particular physical description.

No single list of unit-symbols will be generally useful, however, unless it is devised mainly in the light of the first interpretation and its implications; for this interpretation, unlike the second one, suggests that considerable economy of thought can be effected if the rules for selecting and using unit-symbols are formulated, not mainly arbitrarily, but in the light of the already established mathematical procedures of physics. As an analogy, consider an earlier time when negative exponents had not yet been defined and there were no rules for operating on quantities such as x^{-n}. If standardization had been the only issue, any one of several definitions agreed upon by a majority might conceivably have done. However, a definition and rules of operation for positive exponents were already available, and so the sensible step was to adopt $x^{-n} \equiv 1/x^n$ as the definition, thus making factors involving negative exponents subject to the same rules of operation as those with positive exponents. Integrative steps of this kind, that effect economy of thought and effort, and simplify learning, are of course a main objective of any science.

In line with these ideas, the writer prepared a 35-page mimeographed document on unit-symbols and sent it to a number of physicists in various types of educational institutions and industrial and Government laboratories. Included in the document was a proposed set of rules applying to such symbols and a list of approximately 500 primary and secondary units, together with the various "abbreviations" for each of them that are in more or less common use. In the light of the various comments and criticisms received, the present report has been prepared. Many of the units listed in the original document have been eliminated, the present Table I being confined to those basic units and combining forms that are believed to be in most common use.

PROPOSED RULES FOR FORMING UNIT-SYMBOLS

A number of the symbols listed in Table I do not conform with the following rules, but are so firmly established that attempts to change them would be futile. Exceptions to rules of language are a common occurrence and provide no good argument against having rules that facilitate a choice between several well-established symbols or the selection of a symbol for a new unit.

(1) *The symbol preferably should consist of two or three letters, and never more than four.* Thus

TABLE I. Proposed symbols for physical units.

Unit or combining form	Symbol	Examples of use; comments
ab-	ab	abamp, abcoul/cm²
acre	acre	acre ft
ampere	amp	amp turn/m
	a	With prefixes: ma, μa
angstrom	A	
are [=10² m²]	are	Use sparingly
atom	atom	atom/gm-awu, atom/mole
atomic mass unit	amu	1 amu = 931 Mev
atomic weight unit	awu	1 awu = 1.0002 amu
atmosphere, standard	atm, A_0	
atmosphere at 45°	atm_{45}, A_{45}	
bar [=10⁶ dy/cm²]	bar	
barn [=10⁻²⁴ cm²]	barn	
barrel	bbl	
barye [=dy/cm²]	···	Use μbar or dy/cm²
bel	bel	
	b	With prefixes: db
billion [=10⁹]	B	Bev
Bohr magneton, electronic	μ_0	$1\ \mu_0 = 9.27 \times 10^{-21}$ erg/gauss
Bohr magneton, nuclear	μ_I	$1\ \mu_I = \mu_0/1836$ erg/gauss
Brinell hardness number	Bhn	
British thermal unit	Btu	Btu/lb F°
bushel	bu	bu/acre
calory	cal	
Calory	···	*See* kilocalory
candle	ca	ca hr, ca/m²
candlepower	···	*See* candle
cent (*acous.*)	cent	
cent (*monetary*)	ct	ct/gal, ct/kw hr
	¢	In crowded tables, etc.
centi- [=10⁻²]	c	cm
centigrade degree (*temp. difference*)	C°	cal/cm sec C°, 100C° = 180F°
centigrade temperature		*See* degree centigrade
centimeter-gram-second unit	cgsu	Use sparingly
centimeter-of-mercury	cm-hg	
centimeter-of-oil	cm-oil	
cgs electromagnetic unit	cgsm	*See also* ab-
cgs electrostatic unit	cgse	*See also* stat-
circular (*adj.*)	cir	cir-mil [=0.7854 mil²]
coulomb	coul	coul/m²
cubic	³	cm³ (never "cc"), ft³/sec
curie	c	mc, μc
cycle	cy	cy/sec [=hz]
	c	Often used with prefixes: kc, Mc
day	day	
deca- [=10]	···	Deprecated
deci- [=10⁻¹]	d	db, dm
degree (*of arc*)	deg	deg/sec (*ang. velocity*)
	°	90°
degree absolute	···	*See:* degree Kelvin; d. Rankine
degree Baumé	°B	
degree centigrade	°C	0°C = 32°F
degree Fahrenheit	°F	32.000°F (*ice point*)
degree Kelvin	°K	273.16°K
degree Rankine	°R	491.69°R
deka- [=10]	···	Deprecated
diopter	diop	
division	div	div/sec, div/μv
dollar	dol	dol/hr, dol/ton
	$	In tables, etc.
dozen	doz	doz/hr
dyne	dy	dy/cm², dy cm (*torque*)
electromagnetic unit	emu	Ambiguous; deprecated
electron (*charge*)	e	
electron volt	ev	1 ev = 1.601×10^{-12} erg
electrostatic unit	esu	Ambiguous; deprecated
erg	erg	erg sec, erg/C°, erg/deg
Fahrenheit degree (*temp. difference*)	F°	Btu/F°
farad	fd	fd/m
	f	With prefixes: abf; μf
foot	ft	ft/sec²
foot-candle [=lu/ft²]	ft-ca	lu/ft² is preferable
foot-lambert [=ca/πft²]	ft-lam	
foot-pound-second unit	fpsu	Use sparingly
fresnel [=10¹² cy/sec]	fr	[=MM cy/sec]
gal [=cm/sec²]	gal	g_n = 980.665 gal
gallon	gal	gal/min
gamma [10⁻⁵ oer]	gamma	
gauss	gauss	
geepound	···	Synonym for slug
gilbert	gil	
gill	gill	
grad [=10⁻² rt. angle]	grad	
grain	gr	
gram (*mass*)	gm	gm/cm³ (*density*)
	g	With prefixes: kg; mg
gram of force	gf, gwt	gf/cm², gwt/cm²
gram atomic weight	gm-awu	
gram calory	···	*See* calory
gram molecular weight	···	*See* mole
grav [=32.174 ft/sec²]	grav	
Hartree unit	hu	
hecto-, hect- [=10²]	h	Use sparingly: hm
hekto-	···	Variant of hecto-
henry	hy	hy/m
	h	With prefixes: mh
hertz [=cy/sec]	hz	hz sec [=cy]
horsepower	hp	hp hr
hour	hr	hr/day
	h	Astron. text and tables: 3^h
inch	in.	in./sec
inch-of-mercury	in.-hg	
inch-of-oil	in.-oil	
joule	j	j/mole deg
Kelvin degree (*temp. difference*)	K°	
	deg	Use when not ambiguous: erg/deg molecule
kilo- [=10³]	k	kcal, kev, kmole
kilogram-calory	···	*See* kilocalory
kilocalory	kcal	
kilogram of force	kgf, kgwt	kgf/m² (*weight-density*)
kilogram-mole	···	*See* kilomole
kilomega- [=10⁹]	kM	1 kMev = 1 Bev
kilomole	kmole	
knot	knot, kn	knot hr, kn hr
lambert [=ca/π cm²]	lam	
large calory	···	*See* kilocalory
light-year	lt-yr	
line	line	line/cm²
liter	lit	
	l	With prefixes: ml, k
lumen	lu	lu hr, lu/watt
lux [=lu/m²]	lux	lux sec
magnetic pole	···	*See* pole
magneton	···	*See* Bohr magneton
maxwell	max	
mega-, meg- [=10⁶]	M	Mev, Mm
megamega- [=10¹²]	MM	
megohm [=10⁶ ohm]	meg, Mohm	meg/m, Mohm/m
meter	m	
meter-candle [=lux]	m-ca	lux is preferable
meter-kilogram-second unit	mksu	Use sparingly
mho	mho	mho/cm
micro-, micr- [=10⁻⁶]	μ	μsec, μv, μg
micrometer	···	*See* micron
micromicro- [=10⁻¹²]	μμ	μμf
micron [=10⁻⁶ m]	μm, μ	
mil [=10⁻³ in.]	mil	
mile	mi	mi/gal
mil-foot	mil-ft	ohm/mil-ft
milli- [=10⁻³]	m	ma, ml (not "cc")
millimicro- [=10⁻⁹]	mμ	
millimicron [=10⁻⁹ m]	mμ	
million [=10⁶]	M	Mgal/day; *see* mega-
minute (*of arc*)	min	min/sec (*ang. velocity*)
	′	In tables, etc.
minute (*of time*)	min	
	^{m}	Astron. text and tables: 5^m
mks electromagnetic unit	mksm	
mole	mole	
molecule	molecule	
month	mo	
myria- [=10⁴]	myria	Use sparingly
neper [=8.686 db]	nep	
newton [=kg m/sec²]	new	new/m², new/m² amp turn
normal atmosphere	···	*See* atmosphere, standard
number	no.	no./hr
ohm	ohm	ohm cm
	Ω	In crowded diagrams, etc.
oersted	oer	
ounce (*force*)	oz	
phon	phon	
phot [=lu/cm²]	phot, ph	
pint	pt	
pole, unit magnetic	pole	dy/pole
poise [=dy sec/cm²]	poise	
pound (*force*)	lb, lbf	lb sec or lbf sec (*impulse*)
pound of mass	lbm	lbm/ft³ (*density*)
poundal	pdl	ft pdl
pulse	pulse	pulse/sec
quart	qt	
radian	rad	rad/sec
Rankine degree (*temp. difference*)	R°	Btu/lb R°
revolution	rev	rev/min, rev/sec ("rpm" and "rps" only in crowded tables, etc.)
rhe [=1/poise]	rhe	

TABLE I.—*Continued.*

Unit or combining form	Symbol	Examples of use; comments
rod	rod	
roentgen	r	mr
rowland	row	
r-unit	...	Synonym for roentgen
rutherford	rd	
second (*of arc*)	sec ″	In tables, etc.
second (*of time*)	sec s	Astron. text and tables: 10^s
slug [=32.174 lbm]	slug	slug/ft³ (*density*)
Siegbahn unit	...	Synonym for x-unit
small calory	...	*See* calory
square	²	in.²
stat-	stat	statamp, statcoul
steradian	srad	
stilb [=ca/cm²]	...	Deprecated; use ca/cm²
thousand [=10^3]	k	kBtu, kft
ton (*force*)	ton, tonf	ton mi *or* tonf mi (*work*)
ton of mass	tonm	tonm mi (*transportation*)
turn	turn	
vibration	vib	vib/sec
volt	v, volt	v coul, volt coul
watt	watt w	watt hr With prefixes: kw
weber	web	web/m²
x-unit	xu	1 xu = 1.0020 mA
week	wk	
yard	yd	
year	yr	

"lu" is preferable to "l" for "lumen." Well-established exceptions to this rule include "gamma" for "gamma," "j" for "joule," and "v" for "volt." If the name of a unit contains only two or three letters, this name, rather than an abbreviation, should be used; examples are "bar," "day," "erg," "lux," and "ohm."

(2) *The symbol preferably should consist of the first two or three letters of the name of the unit.* This facilitates recognition and pronunciation, as in "dy" for "dyne," "lu" for "lumen," and "oer" for "oersted." Among the firmly established exceptions to this rule are: "ct," "ft," "hr," "hy," "lb," and "oz."

(3) *Preferably there should be only one symbol for a particular unit.* However, it seems desirable to recommend certain exceptions; for example, both "a" and "amp" for "ampere," and "b" and "bel" for "bel," the "a" and "b" to be used only with combining forms, as in "ma" and "db."

As the rule implies, the same symbol should be used for both singular and plural forms; thus, 10 amp, *not* 10 amps.

One of several objections to such symbols for secondary (derived) units as "fps" and "rpm" is that the "f," "s," "r," and "m" are used in place of the well-established "ft," "sec," "rev," and "min."

(4) *Periods should be omitted from symbols.* An exception is "in.," for "inch," since omission of the period would often result in confusion.

(5) *The symbol for a combining form denoting a multiple or submultiple should be a single letter.* Thus "μ" for "micro-," "M" for "mega-," and so on.

USE OF UNIT SYMBOLS IN PRINTED TEXT

(i) *Set the symbol for a unit in roman type.*

(ii) *Use an abbreviated unit-symbol only when it:* (a) *is preceded by a numerical value, or* (b) *appears in headings of tables or in crowded text, in which cases the symbol is enclosed in parentheses.* Thus: "25 cm"; "several centimeters"; "volumes, in cubic centimeters"; "volumes (cm³)"; "*v* (cm³)."

(iii) *Use standard signs to indicate all mathematical operations with unit symbols.* Thus indicate: (*a*) multiplication by a space, a center dot, or even a ×; (*b*) division by a solidus, a negative exponent, or the ordinary fractional form; (*c*) a power by a positive exponent; (*d*) a root by a fractional exponent or √. Thus dy cm, dy·cm, or even dy×cm, but *not* dy-cm; ft/sec, ft sec^{-1}, or $\frac{ft}{sec}$, *not* fps, fs, or (in equations) ft per sec; cm^3, *not* cu cm or cc.

A hyphen should not be used to indicate multiplication, or a "p" to indicate division, since these devices are not used elsewhere in mathematics for these purposes. A hyphen should be used only when it is desired to separate parts of a single unit symbol; thus a hyphen is appropriately used in "ft-ca," since the "foot-candle" does not represent the product of "foot" and "candle."

One report on "Abbreviations for Scientific and Engineering Terms"[1] advocates abbreviations such as the following for secondary units: cfm (*for* ft³/min); kgps (*for* kg/sec); mphps (*for* mi/hr sec); psf (*for* lb/ft²); rpm (*for* rev/min).

[1] American Standards Association, document Z10.1-1941 (American Society of Mechanical Engineers, 1941).

A form such as "rpm" may be used on crowded diagrams or apparatus labels; but from the standpoint of suitability for substitution in physical equations and ease of mathematical manipulation, the aforementioned forms are comparable to such unconventional and ambiguous algebraic expressions as: *cab* (for a^2/b); *apb* (for a/b); *apbpc* (for a/bc); *asb* (for a/b^2); and so on. Of course, forms such as "cfm" or "mphps" may be regarded merely as special names for secondary units, in the sense that "dyne" and "erg" are names for such units. But the suggestion that special names, many of them unpronounceable, be given to all commonly used secondary units, metric and English, multiple and submultiple, is certainly not a move in the direction of greater economy of thought and learning.

This proposal has been submitted to the recently reorganized Committee on Terminology, Symbols, and Abbreviations of the American Association of Physics Teachers. Interested readers are urged to send criticisms and suggestions concerning the present recommendations to the committee. The committee consists of: J. D. Elder, H. K. Hughes, T. H. Osgood, D. Roller (chairman), L. D. Weld, and M. W. Zemansky. It is contemplated that this committee will be assisted by a group of consultants, to be selected from among those who have an evident interest in contributing to its work.

Teaching a Philosophy of Experimentation in a Course in Electrical Measurements*

FRANCIS T. WORRELL
Rensselaer Polytechnic Institute, Troy, New York
(Received September 12, 1952)

A course in electrical measurements is described wherein the emphasis is shifted from the learning of various electrical measuring techniques to the development of an appreciation of experimental problems in general. This is accomplished by dropping the requirement of written reports, by allowing sufficient time for the performance of each experiment for thoughtful work, and by selecting the experiments for certain features of general experimental significance rather than for specific electrical measuring techniques. Examples are given of typical experiments.

THE course in electrical measurements for physics students contains a number of pitfalls for the unwary instructor who is called upon to organize it. Not the least of these pitfalls is the temptation to start by gathering a large number of resistance boxes, galvanometers, potentiometers, bridges, and other standard electrical measuring instruments; then to select from the numerous textbooks and manuals available a number of standard experiments, and have the student run through them—the more experiments the better. At the end of a course of this sort the student has probably acquired a certain, possibly considerable facility with electrical equipment, and has learned a number of specific techniques. But if he has acquired no more than this, have we accomplished a desirable result? Have we done any more than a good trade school could have done? In other words, have we contributed to the "liberal" scientific education of the student? If the answer to these questions is in the negative—and that would be the author's answer—then we should try to see what more could or should be accomplished in such a course.

The subject material in a course in electrical measurements is relatively easy to grasp. The experimental conditions in most of the standard experiments are generally rather good (compared, for instance, with conditions prevalent in acoustical or thermal measurements), and the techniques and instruments are well developed. A facility with the instruments is not difficult to acquire; further, electrical measurements are made so often in actual work in the research laboratory that the student, or graduate, will have ample opportunity to get the facility later. An appreciation of the potentialities and limita-

* Presented before the American Association of Physics Teachers, Iowa City, June 12, 1952.

tions of different experimental techniques is less easily acquired, however, and is therefore worth more emphasis. If a student performs a number of well-defined experiments with almost sure results he may learn little about these matters, or about the philosophy of experimentation in general; and, after all, this really should be one of the principal objectives of our laboratory courses.

The student taking a laboratory course in a subject which is somewhat difficult to understand exerts most of his effort in the comprehension of the subject; but when the subject material is not difficult the instructor has the opportunity to acquaint the student with some of the mechanics of performing the experiments themselves, and to get him to appreciate more fully the reasons for the methods which are used, and some of the limitations of experimental work. The way to do this, in the opinion of the writer, is, in the first place, to plan to do only a small number of experiments, allowing plenty of time for each one, and, in the second place, to select as far as possible experiments which illustrate *general* experimental procedures rather than *specific* electrical measuring techniques. A start has been made in this direction in the course which will now be described.

The course in Electrical Measurements, a first-term junior course for students majoring in physics at Rensselaer Polytechnic Institute, is a two-credit course consisting of two three-hour laboratory periods a week.[1] The first hour of the first period of each week has been used generally for a discussion of some basic ideas, and of some experimental methods not covered in the laboratory which are considered worthy of special mention. No attempt is made to cover the field in the detail in which a textbook would, nor do the discussions follow the textbook used in the course. Rather, it is hoped that the student will get the general picture from his reference reading for the experiments which he performs.

Reports on the experiments are not required. It is felt that the writing of reports, important as it is—and poorly done as it is, after all the practice which students get—would be done at the expense of the thorough experimentation which this course emphasizes. Instead, the student is asked to keep what the instructor hopes may come to bear a reasonable resemblance to a research notebook. The student is asked to use a bound notebook, since the looseleaf notebook, with all its convenience, encourages the discard of sections containing information thought to be incorrect, but sometimes later found to be of interest. He is encouraged to use his book not only as a place in which to record the usual data, but as a place in which to record passing observations made during the course of the experiment, notes on changes in the originally proposed procedure, and such things; in other words, comments which indicate that he was thinking while he did the experiment. The student is asked to record data directly in the book as he proceeds, even if it means crossing out some mistakes and starting a tabulation over, rather than to take the data on loose pieces of paper with the idea of sifting these data before putting them in the book. The instructor freely admits to them that the book that results may not be as pretty in the first case as in the second, but points out that there may be some valuable information lurking in the unwanted data (the old story of the Ugly Duckling) which would be lost in the selection process. The student who wants a neat book will probably make more of an effort to use foresight rather than hindsight.

In preparation for a given experiment before performing it in the laboratory the student is expected to read the instruction sheet[2] for the particular experiment, also the reference material listed there, and then write in the notebook a brief prospectus including diagrams, procedure, and equations to be used. (This material is not referred to as "Purpose" and "Procedure," because this induces a conditioned reponse in the student, and he sees in his mind's eye a long series of reports reaching back into the past. We want to avoid an association with what may seem to be a routine report which, if filled out

[1] This has been preceded by a four-semester course in general physics, and is concurrent with a course in circuit theory and a course in electric and magnetic fields.

[2] In the past the instruction sheets have been quite complete in themselves, as far as instructions were concerned, but a recent conversation with one of my colleagues has led me to believe that fewer instructions in advance would make for more careful planning on the part of the student. The detailed instructions could be available at the site of the experiment. In case there is no good reference, the sheets must still be detailed.

in the proper way, without need for thought, will pass.) It is emphasized at the beginning of the term that he is merely doing what anyone should do in preparation for tackling an experimental problem

When the student arrives at the laboratory his preparation is evaluated by an inspection of the prospectus which he has written in his notebook. Suggestions are made at that time for improvements, and a grade is given. This grade will be considered in giving a grade on the report.

Also considered in the over-all grade is the performance of the student in the laboratory. The instructor observes his general conduct, the efficiency with which he performs the experiment, his ability to observe experimental difficulties along the way, etc. Occasionally a student will be asked a question during the experiment, usually on some point which might be overlooked by the thoughtless student.

The student is asked to make some semi-quantitative statement as to the precision of the result. In his thinking he must consider (1) instrumental and, occasionally, statistical errors; (2) observational errors resulting from limitations in the readability of the instruments; (3) the sensitivity of the method. In the first item the students have already been drilled in their general laboratory, but in the others they have had no practice, and their evaluation requires more mature thought.

When the student has made the measurements, he of course calculates the results and evaluates them, including in his evaluation consideration of the precision, as mentioned above. He is encouraged to respect his results, and not to worry about the "correct" value. In cases where he measured an "unknown," it remains "unknown," except in so far as he believes his results. He is not told what he should get. In one experiment, where the dielectric constant of air is measured, he does have a handbook value with which to compare his results. Since the results he gets are generally in disagreement with the accepted value he is asked to see if he can explain the deviation in terms of certain experimental conditions.

All this is intended to encourage him to take the attitude that his meters have told him something, and that it is his responsibility to decide *what* they have told him. Have they told him what he thinks they did, or something else; that is, is the experiment valid? If it disagrees in its results with some other determination this does not mean that the student's work is "wrong," but that the underlying experimental conditions were not suitable. The student is allowed one week for each experiment, which means five or six hours in the laboratory, depending on whether some time is used for general class discussion. The experiments themselves are mostly quite standard, but it is the way in which they are used which the author wishes to emphasize.

In illustration of some of the points which have been discussed so far, let us look at some of the experiments which are performed in the laboratory, confining our discussion only to the particular features which are illustrative. The measurement of resistivity using the Kelvin double bridge provides two points of interest. First, the student, if he evaluates his precision correctly, will find that the principal source of error is not the measurement of the length of the test bar (about 1 meter) with a meter stick but rather the measurement of the diameter (about $\frac{1}{4}$ inch) with a micrometer. Second, the good student may have some time to check his measurement using a type-K potentiometer. He soon comes to realize that although the type-K is potentially more precise than his bridge, his results obtained with the potentiometer are poorer because of unsteady current through the unknown resistance.

In one experiment a Weston Model 45 ammeter is calibrated against a Leeds & Northrup Student potentiometer. The student is asked whether it would have been better to calibrate the meter using (a) another Model 45 meter in series with it, or, (b) a type-K potentiometer.

In another experiment a 1-μf condenser is measured in a Wien bridge, then in a Maxwell bridge. While the use of the latter to measure capacitance in terms of inductance is admittedly the reverse of the usual procedure, no harm seems to have resulted. The use of the two methods provides the student with the opportunity of comparing different methods of making the same measurement. He is asked to compare the two methods with regard to the standard

used in each case (a stepwise-variable capacitor *vs* a continuously variable inductor), the ease of balancing, and the precision. In connection with the last item, he is asked to determine the sensitivity of the bridge, i.e., the minimum detectable unbalance. He is also shown how to improve his balancing procedure by interpolating between two points of equal unbalance, rather than by trying to find the null directly.[3] For detectors we firmly stick to the headphone, praising the detecting qualities of the ear. This is sometimes over the loud protests of the gadgeteers who demand nothing less than an oscilloscope.

In another bridge experiment the frequency bridge is used to measure the variation of inductance of a variable inductor with frequency. The student is to plot his results in the form of curves of apparent inductance *vs* frequency for three chosen scale readings of the inductor. Since the capacitor used as a reference in this bridge has a stepwise adjustment it is not possible, in general, to achieve a balance at the chosen scale setting. Instead, one must plot inductance as a function of scale setting for several different frequencies, then from these curves obtain data from which to plot the desired curves. This is an important technique for them to know.

In the measurement of the volt-ampere characteristics of rectifiers several are used, including a vacuum diode. A test of the student's alertness is to see whether in the case of the diode he records unthinkingly that zero applied voltage gives zero current. You'd be surprised how many do!

In the calibration of a Hewlett Packard audio oscillator the student is presented with the difficulty that the only available laboratory standard is another Hewlett Packard oscillator no more precise than the one to be calibrated. This difficulty is resolved by setting the reference oscillator to zero beat with the 440-cps note from radio station WWV, then calibrating the unknown in the range 44 cps to 4400 cps by using Lissajous figures on the screen of an oscilloscope. To extend the range of calibration upward the unknown is set with care at 4400 cps, the reference brought up that frequency, and the calibration continued. This equipment thus exhibits two techniques of value. The latter technique, for instance, has been used in the calibration of an optical pyrometer.

The six experiments which have just been discussed represent, of course, only part of the list. The intent has been not to present a detailed discussion, but to point out the aspects of some typical experiments which are considered of importance in the course as we teach it.

In summary, the laboratory in electrical measurements which has been described is intended to give the student an appreciation of the general problems of experimentation, rather than a knowledge of specific techniques. To this end ample time is given for the performance of each experiment, and the general nature of the experimental method is emphasized. Experiments are chosen on the basis of their general experimental value rather than on the basis of any specific technique.

[3] In this connection we can point out the applicability of this technique to such things as photometric measurements.

About 1904 I was asked by Rutherford to make a sensitive small-capacity electroscope the gold leaf of which would remain charged for two or three days. This I failed to do. So Rutherford said: "Lester Cooke used to make them; why can't you? Get Jost the mechanic to make you one!" So I went to Jost and repeated this. He said: "If I could not make a better electroscope than Cooke, I'd shoot myself!" So he made a beauty to look at, but a bad one to go. Its leaf collapsed in twelve hours. This puzzled me. One night I could not sleep and got up in my diggings and made an electroscope of a tobacco tin, an amber mouthpiece of a tobacco pipe, and some Dutch metal foil; charged it with sealing wax and went to sleep. The leaf of this home-made freak electroscope remained open and charged for three days, and solved the problem. An electroscope made of material outside the laboratory would remain charged for 48 hours inside the laboratory, but all the material inside the building was contaminated and coated with active deposit including the slow period transformations of radium. . . . Precautions were then taken to prevent the escape of radon which had been at the root of the mischief, hitherto unsuspected.—A. S. Eve.

Experimental Reactor Physics Course at the Oak Ridge School of Reactor Technology

E. D. KLEMA, R. J. STEPHENSON,* AND S. TAYLOR
Oak Ridge National Laboratory, Oak Ridge, Tennessee
(Received August 27, 1952)

The curriculum of the Oak Ridge School of Reactor Technology includes a laboratory course in reactor physics. The first experiments are concerned with the fundamentals of nuclear physics and the later ones are of direct relevance to reactor engineering. The laboratory manual of the course is available as AECU-2164.

THE Oak Ridge School of Reactor Technology (ORSORT) presents a year's basic training in reactor engineering. The curriculum consists of about 38 weeks of classroom work including 26 weeks of laboratory, followed by 12 weeks of either in-service training or participation in a reactor design study. Lectures are presented in reactor theory, mathematics, reactor materials, reactor physics, reactor chemistry, health physics, reactor technology, and reactor engineering. Students are either on leave from industries associated with The Atomic Energy Commission's work or are recent graduates in physics, chemistry, or any of the various branches of engineering.

The large variation in the scientific backgrounds of the students presents peculiar educational problems. In particular, the usual engineer has little or no knowledge of nuclear phenomena. Thus the first few experiments in the laboratory and some of the lecture material in the reactor physics course are not essentially new to most of the physicists and chemists. However, since the majority of students are engineers, it is necessary at present to include this material in lectures and laboratory. It is hoped that in the future much of the introductory laboratory material can be included in the students' undergraduate training.

The laboratory course at ORSORT is the product of the work of a number of people at Oak Ridge National Laboratory who have designed the experiments and instructed in the course. In particular, we should like to acknowledge the work of Dr. E. C. Campbell who initiated this laboratory course in 1950. At the beginning of the training period in September the students are given a set of instruction sheets covering the experiments for the year. These contain references, the pertinent theory and procedure of the experiment, and questions for discussion. It is neither desirable nor possible to give anything but a brief account of the experiments in this article. However, anyone who is further interested in these experiments may purchase a complete set of instruction sheets for the 1951–1952 course.[1]

During the year the student performs some 26 experiments. The first group of these deals with alpha- and beta-particles, gamma-rays, Geiger counters, simple and complex radioactive decay, and health physics measurements. Most of these experiments are or could be performed in any modern physics course. Following these are five experiments in chemistry and the remainder are largely group experiments performed at the Oak Ridge graphite reactor. The student body is broken into four laboratory groups of approximately 18 each. Wherever possible, the students work in pairs on individual pieces of apparatus, though for many experiments associated with the reactor this is not possible.

The instructor holds a one-hour discussion on each experiment before it is performed. Questions on the previous experiment may also be discussed at this time. After the experiment is performed the student hands in a report consisting of the instruction sheets, the original data taken in the experiment together with appropriate calculations and graphs, a discussion of the errors involved in the experiment, and answers to the questions. After the first experiment has been

* On leave from The College of Wooster, Wooster, Ohio.

[1] AECU-2164, "A Laboratory Course in Reactor Physics," Office of Technical Services, Department of Commerce, Washington 25, D.C.; price $1.05.

completed, including the report, each student is given a copy of a sample report of this experiment made by an instructor. This is done to indicate to the student the degree of completeness and detail required in the reports. In general, this has proved a very useful device both for the student and for the instructor.

The first experiment is given largely to acquaint the student with the build-up and decay of radioactive nuclei produced by neutron bombardment of a small piece of silver or indium. The activation is done in a "neutron howitzer" consisting of a Po-Be neutron source, emitting about 10^7 neutrons per second, placed at the center of a cubical paraffin block two feet on edge. By placing the sample about one inch from the source an irradiation time of ten minutes sufficiently activates the sample so that it can be counted with the usual Geiger counter and scaling equipment.

The subjects of the next three experiments are the specific ionization due to alpha-particles, the absorption and range of beta-particles, and the absorption of gamma-rays. In the alpha-particle experiment the ionization produced in a shallow ionization chamber is measured by a vacuum tube electrometer as a function of the distance from the alpha-source. The results are then plotted, giving a Bragg curve. The beta-particle experiment is concerned with measuring the intensity of beta-particles from P^{32} after passing through various thicknesses of aluminum. An end-window Geiger counter is used to measure the intensity of the beta-rays. By plotting the logarithm of the intensity against the absorber thickness the maximum beta-particle range is found from the curve. From the empirical relationship between the maximum energy and range given by Glendenin, the maximum energy of the beta-particles emitted by P^{32} is determined. In the next experiment the absorption coefficients of gamma-rays from a Co^{60} source are measured for lead and steel. The gamma-ray intensity transmitted through various thicknesses of lead and steel is measured by means of a Lauritsen electroscope. Absorption coefficients are obtained from the slopes of the semilog plots of intensity against thickness of absorber.

The next experiments are concerned with typical health-physics measurements. In the first of this group of three experiments, the students calibrate a beta-gamma survey meter, using a calibrated radium source. The dose rate in milliroentgens per hour is calculated for various distances between the source and survey meter and the corresponding meter readings are taken. A curve is then plotted of meter readings against the dose rate for each of the three scales on the instrument. The beta-gamma survey meter consists of a portable Geiger counter and appropriate electronic equipment using a count-rate circuit having three scales corresponding to maximum doses of about 20 mr per hr, 2 mr per hr, and 0.2 mr per hr. The second experiment consists in determining the amount of radium in an unknown source by comparing it with a known source. A Lauritsen electroscope is used as the measuring device. In the final experiment in this group the students use a variety of instruments to survey the gamma-ray flux and the fast and slow neutron fluxes around the Oak Ridge graphite reactor. An ionization chamber type of instrument called a "cutie-pie" is used to measure gamma-ray intensities. For the thermal neutron flux a pair of Lauritsen electroscopes is used, one of which has its chamber coated with boron. In general, measurements are made of thermal neutron fluxes in the presence of a fairly large gamma-flux. The difference in the readings of the coated and uncoated instruments gives a measure of the thermal neutron flux. A calibration curve is supplied with the instruments so that it is possible to measure thermal neutron fluxes from zero to twice the maximum permissible dose of 1750 thermal neutrons per cm^2 per sec. The fast neutron flux instrument consists of a paraffin-lined proportional counter in which the proton recoils from the paraffin are measured. This instrument is designed to detect and measure approximately the fast neutrons in the 0.2 to 10 Mev energy range and from 10 to 10 000 fast neutrons per cm^2 per sec.

In the next experiment the students have their only experience during the course with electronics as such. They wire a Higinbotham scale-of-two circuit and test it for correct operation, examining operating voltages and wave forms of the voltage at various points in the circuit. Finally the individual units are

placed in series to simulate a scaling unit of 512, 1024, etc.

The experiment on Geiger-Mueller counters is designed to familiarize the students with some of the characteristics of these counters and also to give a brief introduction to the statistics of counting. Natural uranium samples are used as sources. The resolving time of the counter and associated circuit is found by the two-sample method. A plot of counting rate *vs* tube voltage is also obtained. Finally, the number of counts occurring in a short time interval is observed over a large number of time intervals, and a statistical analysis is made of the observed distribution.

In the experiment on the analysis of a complex radioactive decay curve, the half-lives and initial activities of two independent radioactive substances are measured. The mixed substances are chosen so that one has a relatively short half-life of a few minutes while the other has a relatively long half-life of the order of an hour. The resultant activity, measured with a Geiger counter and scaler, is plotted on semilog paper against the time after removal of the sample from the reactor. By subtracting ("peeling off") the activity due to the long half-life substance from the total activity, the activity of the short half-life substance is found.

The next five experiments are concerned with chemical techniques and processes of importance to reactor engineers. The first of these involves the separation of metallic elements by solvent extraction. A single-stage water-benzene extraction is used to separate a mixture of aqueous zirconium and thorium perchlorates. Radioactive tracer techniques are used for following the thorium and zirconium through the extraction cycle. In the next experiment, an ion-exchange separation of cobalt and nickel is performed. An ion-exchange column containing a strong base-type anion resin is used. The cobalt is followed with a radioactive tracer, while the presence of nickel is determined by chemical test. The third experiment in this group is concerned with radioactivation analysis. The manganese content of an aluminum alloy sample is determined, first, by calculation, using the known neutron flux, cross section, etc., and the measured activity of the sample, and, second, by comparing the activity of the alloy sample with that of a sample of pure Mn. In the first method, a calibrated high-pressure gamma ionization chamber is used for obtaining the sample activity. In the latter method the samples are dissolved, then evaporated on watch glasses in a standardized manner, and counted with an end-window Geiger counter. The next experiment deals with radiation chemistry. The build-up of pressure over a distilled water sample is observed as a function of time of exposure to gamma-rays from a Co^{60} source. The experiment is then repeated using a water solution of potassium bromide. Next, a crystal of sodium chloride is irradiated by gamma-rays at liquid nitrogen temperature. The crystal is allowed to warm slowly, and the release of stored energy in the form of visible light is observed by means of a photomultiplier tube and associated circuit. The final chemistry experiment consists of determining the neutron flux in the Oak Ridge graphite reactor by measuring the amount of fission product Ba^{140} in a natural uranium sample that has been irradiated in the reactor for a known time. The radioactive barium is separated from the uranium chemically, using natural Ba as a carrier. The amount of Ba^{140} present is determined by counting with a calibrated end-window Geiger counter. As the fission yield, decay constant of Ba^{140}, and the U^{235} fission cross section are known, the neutron flux can then be calculated. The flux is also measured by irradiating an aluminum alloy sample of known Mn content. The activity of the sample is determined with a calibrated high pressure gamma ionization chamber, and the neutron flux computed.

For the experiment on neutron scattering and absorption, the neutron howitzer is used together with boron-trifluoride proportional counters. The back scattering of neutrons by paraffin blocks is measured as a function of the number of blocks, while the total cross section is measured for aluminum, iron, and paraffin by the transmission method. Relative scattering cross sections of carbon, aluminum, and lead are determined by scattering a beam of thermal neutrons from the graphite reactor through 90° by the different substances and measuring the scattered beam with a BF_3 proportional counter.

The experiment on the measurement of thermal neutron capture cross sections gives the students an opportunity to operate the "pile oscillator," a piece of research equipment regularly used at the Oak Ridge graphite reactor. With this apparatus the students measure the thermal neutron capture cross sections of various materials relative to that of gold. They also measure the absolute "capture areas" of various cadmium and boron samples.

In the next experiment the students obtain beams of monochromatic neutrons by crystal diffraction. By means of transmission measurements a set of boron absorbers is calibrated for use in a later experiment. Since a beam hole at the Oak Ridge graphite reactor is available to the school, it is possible to set up a simple neutron diffraction apparatus for the exclusive use of the students instead of having them perform the experiment on apparatus ordinarily used for neutron diffraction research. In this manner the essential features of the apparatus are made clear at a not too serious sacrifice in the quality of the data obtained.

The calibrated boron absorbers are used in the next experiment to measure the neutron energy at which the main resonance in indium appears. This is done by measuring the activations of indium foils by a cadmium-filtered neutron beam from the Oak Ridge graphite reactor with and without one or more of the boron absorbers. Use is made of the fact that the absorption cross section of boron varies as the reciprocal of the neutron velocity.

For the experiment involving measurement of the number of neutrons emitted per thermal fission and the thermal fission cross section of U^{235} the students are given no detailed instructions. Rather they are provided with the equipment necessary for the performance of the experiment, and are expected to work out the detailed experimental procedure themselves. The equipment and facilities provided consist of a U^{235} sample, a collimated neutron beam from the Oak Ridge graphite reactor, a calibrated parallel-plate fission chamber and associated equipment, a calibrated hydrogen-recoil fast neutron counter and associated equipment, cadmium shields, indium foils, a "standard pile" in which the neutron flux is known, and an end-window Geiger counter.

In the next experiment the students measure the half-lives associated with the delayed neutrons emitted following thermal neutron induced fission in U^{235}. This is done using the "fast pneumatic," a piece of research apparatus at the Oak Ridge graphite reactor which enables one to remove a sample from the reactor and to start counting it in a fraction of a second. The delayed neutrons are detected with a suitably shielded BF_3 counter, and the apparatus plots directly the logarithm of the counting rate as a function of time. From this record the students find the half-lives of the various neutron groups using the "peeling off" technique learned in an earlier experiment.

Next the students measure the diffusion length of thermal neutrons in graphite by exposing a series of indium foils along the vertical axis of a rectangular parallelepiped of graphite which contains a Po-Be neutron source. The saturated activities of the foils are computed for distances from the source at which the neutron flux is nearly pure thermal, and the logarithm of the saturated activity is plotted against the distance from the source. From the slope of the straight line thus obtained and the known dimensions of the graphite block the diffusion length is calculated.

In the next experiment, on the moderation of neutrons in graphite, a graphite stack containing a Po-Be neutron source is used. Cadmium-covered indium foils are inserted in the stack at various distances from the source. The foils are counted with Geiger counters and scalers, and the Fermi age of the neutrons of In resonance energy is then obtained from a curve of logarithm of saturated foil activity *vs* the square of the distance from the source.

The next experiment consists of a study of the neutron flux distribution in a subcritical reactor or "exponential pile." The pile used is an eight-foot cube duplicating the lattice of the Oak Ridge graphite reactor. This cube rests on a graphite base containing a Po-Be neutron source. Traverses are made vertically and horizontally using BF_3 counters. (The horizontal traverse is made parallel to the fuel channels.) A Cd-

covered rod, simulating a control rod, is then inserted vertically in the pile and the horizontal traverse repeated. The material buckling of the lattice is determined from the data of the vertical traverse, and is then used to estimate the size that the pile would have to be in order to be critical.

The flux distribution perpendicular to the fuel channels in the Oak Ridge graphite reactor is studied in the next experiment. One end of a long copper wire is attached to a small block or "rabbit" which fits snugly into a tube running into the pile. The rabbit is drawn into the tube by the partial vacuum maintained in the reactor. After a suitable exposure time, the wire is reeled out. The end of the wire is then soldered to another copper wire which passes through a lead pig and on to a take-up reel which can be driven at constant speed by a motor. The lead pig contains a beta-proportional counter whose output is connected through an amplifier to a count-rate meter and speedomax recorder. After waiting about an hour for the decay of the short lived 5.1-minute Cu^{66}, the motor is started and a record of the activity of the wire as a function of its position in the pile is thus recorded by the Speedomax.

The final experiment of the course at present is a study of the transient behavior of the Oak Ridge graphite reactor during shutdown and start-up. The logarithm of the reactor power and the instantaneous reactor period are recorded graphically from the neutron-induced ionization in a gamma-compensated chamber. From the measured period on start-up the multiplication factor, k, (the factor by which the neutron flux changes from one neutron generation to another) is calculated for the supercritical reactor. After shutdown, the delayed neutrons constitute a neutron source and this enables one to calculate the multiplication factor of the subcritical reactor. The temperature coefficient of reactivity is also calculated from a temperature chart of a uranium slug.

This year it was found possible to have students actually operate a reactor. The student body was divided into groups of six and each group spent two hours at the low intensity research reactor at Oak Ridge National Laboratory. This formed a suitable culmination to both the lecture and laboratory courses.

The Steady-State Theory

The expansion of the universe, which can be inferred either from thermodynamics or from astronomical observations, would seem to lead to a thinning out of material. By the perfect cosmological principle the average density of matter must not undergo a secular change. There is only one way in which a constant density can be compatible with a motion of expansion, and that is by the continual creation of matter. Only if the diminution of density due to the drift to infinity is counteracted by a constant replenishment of newly created matter can an expanding universe preserve an unchanging aspect.

There is little doubt that the continual creation of matter necessary in this [steady-state] theory is the most revolutionary change proposed by it. There is, however, no observational evidence whatever contradicting continual creation at the rate demanded by the perfect cosmological principle. It is easily seen that this is, on the average,

$$3\times(\textit{mean density of matter in the universe})\times\textit{Hubble's constant}=10^{-43}\ \mathrm{g/cm^3\ sec}$$

approximately. In other words, on an average the mass of a hydrogen atom is created in each liter of volume every 10^9 *years. . . . It is clear that it is utterly impossible to observe directly such a rate of creation. There is therefore no contradiction whatever with the observations, an extreme extrapolation from which forms the principle of conservation of matter.*
—*Cosmology*, by H. Bondi, Cambridge University Press, 1952, p. 143, by permission.

Surface Energy, A Mode for Energy Absorption During Impact

JOHN S. RINEHART
Michelson Laboratory, U. S. Naval Ordnance Test Station, Inyokern, China Lake, California
(Received July 14, 1952)

The role that pulverization of the target material may play in absorbing the energy of an impacting missile is discussed. The energy that can be absorbed in this way will, in general, depend on the area of new surface that is created. Approximate calculations suggest that, for impacts against rock-like materials and glass, very considerable amounts of energy could be absorbed. The production of rock flour at the Barringer Meteorite Crater in Arizona is cited as an example.

MANY processes can be operative during conversion to a stable situation of the transitory energies of impacting missiles and of exploding charges. It seems not unlikely that, in some cases, pulverization of a portion of the target material by the impact of the missile or the force of the explosion will play an important role. Such, of course, would not be the case for a perfectly plastic impact, *viz.*, a steel bullet striking a lead target. It is well known, however, that glass and glass-like materials are broken up into very fine flour-like material when subjected to shock loading.[1] The crushing of glass under explosive attack has been discussed recently by Pugh and his co-workers[2,3] and by Kolsky and his co-workers.[4,5] The Barringer Meteorite Crater in Arizona contains millions of tons of rock flour. Barringer[6] has pointed out that it is almost certain that this rock flour is sandstone that was pulverized at the time the meteorite impacted. Considerations that will be discussed here suggest that in the foregoing cases, storage of energy on the fracture surfaces of the comminuted material may be an important way by which energy is absorbed by the target.

The amount of energy E_s that can be converted by this process will depend upon (1) the total amount of new surface that is generated by the impact of the blow, and (2) the energy that is required to form each unit area of this surface. Analytically, these facts may be expressed by the equation

$$E_s = A\sigma,$$

where A is the total surface area and σ is the energy stored per unit surface. In the case of liquid droplets, σ is equal very nearly to the surface tension of the liquid.

It is convenient physically to think in terms of the mass of material that will be pulverized. If a is taken as the new area generated per gram of material comminuted, the energy stored on the surface of a gram of material will be given by the equation

$$e_s = a\sigma.$$

If now we assume that a projectile of mass m and velocity v strikes a target, the mass M of target material that will be crushed can be assumed to be given roughly by the equation

$$M = \left(\frac{\frac{1}{2}mv^2}{e_s}\right)K,$$

where K is a factor that represents the efficiency of the conversion of the kinetic energy of the projectile to surface energy. Absorption of energy by the projectile itself will not be considered here although it could play an important role in many cases.

The primary problem in deciding how important comminution may be as an energy absorbing process is, from a physical point of view, one of evaluating e_s and K. First, let us consider the factors upon which K depends. Some 80 years

[1] John S. Rinehart, Meteoritical Society, Popular Astron. **58**, 9 (November, 1950).
[2] Pugh, Heine-Geldern, Foner, and Mutschler, J. Appl. Phys. **22**, 487 (1951).
[3] Pugh, Heine-Geldern, Foner, and Mutschler, J. Appl. Phys. **23**, 48 (1952).
[4] Kolsky, Snow, and Shearman, Research **2**, 89 (1949).
[5] D. G. Christie and H. Kolsky, Trans. Soc. Glass Technology **36**, 65 (1952).
[6] D. M. Barringer, Paper read before National Academy of Sciences at its Autumn Meeting at Princeton University, November 16, 1909.

ago, Rittinger[7,8] stated that the useful work accomplished in crushing and grinding of ores is directly proportional to the new surface area produced. Most investigators who have discussed comminution problems since that time have assumed that most of the energy goes into heating up the rock and that only a few percent goes into breaking it up. Thus in normal crushing operations, the amount of energy stored on the surface is small compared with the work done in crushing the material. However, it seems not unlikely that during high speed impact or during impulsive loading of the type produced by the detonation of an explosive charge, a much larger fraction of the energy will go into surface energy. It is apparent that K will depend strongly on the physical properties of the target material. If the target material flows plastically, then K might very well be equal to zero. It is to be remembered in this connection that the target material may behave very differently under the extreme dynamic loading conditions that exist during comminution than it would under static loading. For materials that seem to behave in a brittle fashion under impact, such as glass and crystalline quartz, K may be quite large. Since in a truly brittle material no plastic flow can occur, K might approach a value nearly equal to one. In so far as the author knows, no quantitative determinations of the value of K have been made so that whatever numerical value is assigned, K is at best an estimate. Indeed certain investigators feel that, under the high pressures created by the impact, glass will flow plastically so that the value of K will be negligibly small. The author does not hold this point of view.

In order to make a realistic estimate of the value of e, two factors must be considered: (1) the energy stored per unit area of surface, and (2) the distribution in shapes and sizes of the comminuted material. The exact value to use for σ in a given case is not clear. In the case of quartz, investigators[9] concerned with comminution problems often take a value of about 900 ergs/cm^2. Griffith[10] used 520 ergs/cm^2 in connection with his studies of fracturing in glass. Recent work[11] in connection with silicosis produced by quartz particles indicates the existence of a highly soluble surface layer on these particles. An experimental determination of surface energy is extremely difficult because a freshly created surface will become contaminated almost at once. In brief, the whole problem of the amount of energy represented by a square centimeter of surface on a quartz or glass particle can be only very roughly approximated. It seems, however, not unreasonable to assume that the energy probably ranges from about 200 ergs/cm^2 to about 900 ergs/cm^2.

Similarly, distribution of shapes and sizes of particles is also not known exactly for particles whose dimensions are less than a few microns. Indeed, accurate size distribution measurements below 10 microns are rarely accomplished and are usually of doubtful validity.[7] The rock flour found in the bottom of the Barringer Meteorite Crater is an excellent example of crushing due to impact. Merrill[12] has described this rock flour as having a sharp gritty feeling when rubbed between the fingers. Under the microscope, the individual particles of the flour are found to be sharp angular bits of quartz that must have come from shattering of individual grains of sand. There are millions of tons of this rock flour in the bottom and along the sides of the crater. We found that about 30 percent of one sample of flour passed a 325 mesh (43 micron) screen. Gaudin[9] estimates that particles as fine as 0.001 micron might be produced by comminution. Rosenthal,[13] in connection with other studies, has recently examined 0.05-micron marble particles that were produced by crushing. As regards surface area per unit mass, Gaudin[9] suggests that the area of an individual particle is likely to be about 1.7 times that of a cube or sphere of equivalent mass.

It is possible to form a very rough estimate of

[7] Fred C. Bond and Jen-Tung Wang, Trans. Am. Inst. Mining Met. Engrs. **187**, Mining Engineering (1950).

[8] Fred C. Bond and F. E. Briber, Jr., Pit and Quarry **44**, 173 (1950).

[9] A. M. Gaudin, *Principles of Mineral Dressing* (McGraw-Hill Book Company, Inc., New York, 1939).

[10] A. A. Griffith, Trans. Roy. Soc. (London) **A221**, 163–198 (1920).

[11] Nagelschmidt, Gordon, and Griffin, Nature **169**, 538–540 (1952).

[12] George P. Merrill, Smithsonian Misc. Coll. **50**, Part 4 (1907).

[13] D. Rosenthal and M. Kaufman, J. Appl. Phys. **23**, 600 (1952).

the amount of material that would be crushed by a particular missile, such as a large meteorite, if we make some very simplifying assumptions. As an example, assume (1) that the target material is broken up into cubes all one micron on a side, (2) that the energy stored on the surfaces is equal to 500 ergs/cm^2, and (3) that K, the conversion coefficient, is equal to one. We find, with these assumptions, that 1.2×10^7 ergs of energy could be stored on the surfaces of one gram of comminuted target material of density 2.5 g/cm^3, i.e., about that of quartz. It is interesting that experimentally[1] it has been found that small caliber ammunition will on impact lose energy at the rate of about 2.6×10^7 ergs per gram of rock crushed.

It is recognized, of course, that the above treatment is neither rigorous nor exact. Nonetheless, it is believed that storage of energy on surfaces of comminuted material may be of much significance as regards the partition of energy when large high velocity missiles such as meteorites form craters and that further pursuit of the problem might be profitable.

In conclusion, appreciation is expressed to Dr. H. H. Nininger who kindly supplied a sample of rock flour from the Arizona crater and to Ensign W. G. Feuerstein who examined the flour.

An Undergraduate Experiment for the Determination of Alpha-Particle Range

WALTER C. MICHELS AND EVA WIENER*
Bryn Mawr College, Bryn Mawr, Pennsylvania
(Received August 8, 1952)

Simple apparatus, using a scintillation counter with a 931-A photomultiplier, is described. The range of alpha-particles from Po^{210} is determined with this apparatus by varying the pressure in a shielded container, keeping the geometry constant. Data are presented on a 0.06 millicurie source, yielding a range of (3.81 ± 0.03) cm at 760 mm Hg and 15°C. This value is obtained after correction for finite source thickness. The experiment offers particular value in giving students an opportunity to consider statistical fluctuations and to analyze the effects of finite source thickness, straggling, etc.

RECENT developments in connection with the simple and low cost scintillation counter[1] make feasible the increasing use of radioactivity and nuclear experiments in the undergraduate laboratory. The apparatus and the experiment described here offer one such possibility.

In order that the range of alpha-particles may be determined with accuracy, it is desirable that the geometry of the source-detector system be kept constant. This is accomplished in the apparatus shown in Fig. 1 by the use of a rigid chamber which may be evacuated to vary the "effective path" of the alpha-particles without displacing either the source or the detector. Since the container has been constructed to provide shielding for a variety of experiments, it is considerably heavier than is necessary for alpha-particle work. It consists of two coaxial cylindrical shells of Cerrobend,[2] separated by a one

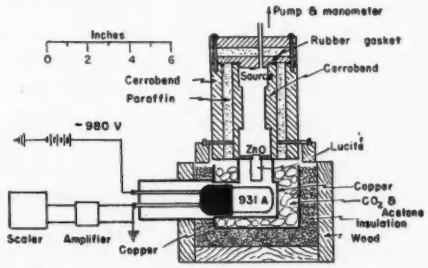


FIG. 1. Apparatus for determination of alpha-particle range.

* Now at Stanford University, Stanford, California.

[1] For a discussion of these developments and an extensive bibliography, see W. H. Jordan, "Detection of Nuclear Particles," Annual Review of Nuclear Science (Annual Reviews, Inc., Stanford, 1952), Vol. 1, pp. 207–226.

[2] Cerrobend is a commercially available material prepared by Cerro de Pasca Company. Its composition is Bi 50.0 percent, Pb 26.7 percent, Sn 13.3 percent, and Cd 10.0 percent. The bismuth and lead offer high mass absorption for beta- and gamma-radiations, while the cadmium guarantees absorption of slow neutrons. The material melts at 70°C, so that it can be cast with great ease. It also machines readily.

centimeter thickness of paraffin. The cover is similarly constructed.

Provision is made, as shown, for cooling the 931-A photomultiplier, which is operated at 980 v and with 10^5 ohms across each stage. Both batteries and a rectifier have been used to supply the potential to the photomultiplier, but it has been found extremely difficult to filter the rectifier output sufficiently to reduce the background count to the level which is achieved with a battery supply. The output of a two-stage *R-C* coupled amplifier, operating at a voltage gain of approximately 60, is fed into a Berkeley scaler. When weak sources are used, the scaler could be replaced by a single decade unit and a high speed mechanical counter.

We have found zinc oxide[3] to be a satisfactory

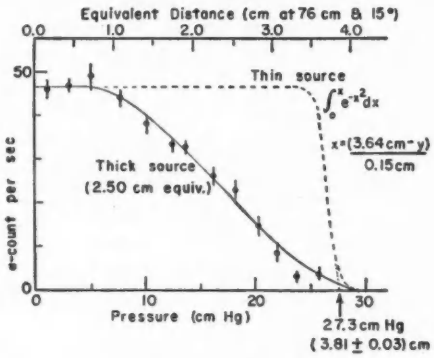


FIG. 2. Analysis of the data.

phosphor for alpha-particle detection. The distance between the polonium foil source and the ZnO is 9.80 cm. The "equivalent distance" in air, under the standard conditions of pressure P_0 and Kelvin temperature T_0 is therefore

$$y=(9.80\text{ cm})(PT_0/P_0T), \tag{1}$$

where P and T are, respectively, the pressure and Kelvin temperature in the chamber. The pressure is varied either by the use of a controlled leak or by variation of the pumping speed. The temperature correction is particularly important when the photomultiplier is cooled. With dry ice and acetone used as a coolant, the chamber of our apparatus runs at -1.5°C.

[3] Fluorescent ZnO supplied by J. R. de Vore of the New Jersey Zinc Company. This is an experimental phosphor labeled CP-215.

With our arrangement, the phosphor subtends an angle of about 7×10^{-3} steradian. The efficiency of the counter, with the "light piping system" shown, is about four percent. The counting rate is therefore approximately 2×10^{-5} of the disintegration rate of the sample. The photomultiplier used in this work, a typical 931-A tube, shows a background count of about 100/sec at room temperature and about 40/sec at dry ice temperature. Taking the minimum practical source as one which will give a counting rate equal to the background of the cooled system at low pressures, we find that this source must exhibit about 10^6 disintegrations per second, i.e., must have an activity of about 0.05 millicurie. The upper limit of source strength is, of course, determined by the resolution time of the amplifier and the scaler.

The experimental points shown in Fig. 2 indicate the net counts observed at various pressures, using a 0.06 millicurie polonium foil source, and the uncertainties as determined from the statistics. The scale shown at the top of the curve represents the reduction of the observed pressures to equivalent distances at 760 mm Hg and 15°C, as determined by Eq. (1).

It is clear that the data shown are typical of a thick source. In order that these data might be reduced to those for a thin source, the following reasoning was used. It was assumed that the straggling of alpha-particles about their mean range R could be represented by a Gaussian curve. The rate at which alpha-particles reach a distance y from a thin source is then given by

$$N_y=N_0\left\{1+(2/\sqrt{\pi})\int_0^x e^{-x^2}dx\right\}, \tag{2}$$

where N_0 is a constant and $x=(R-y)/\alpha$, α being a straggling constant. Let us now suppose that there is a uniform distribution of radioactive material between the surface of the foil and a depth Z, measured in equivalent air distance, below the surface. Then the rate at which particles leave the layer at depth between z and $z+dz$ from the surface is

$$dN_2=(N_0/Z)dz.$$

The rate at which particles starting from this layer reach a point at a distance y from the

surface is then

$$dN = dN_2\left\{1+(2/\sqrt{\pi})\int_0^{x-z/\alpha} e^{-(x-z/\alpha)^2}dx\right\}.$$

Hence the counting rate at a distance y is

$$N=(N_0/Z)\int_0^Z dz \times\left\{1+(2/\sqrt{\pi})\int_0^{x-z/\alpha} e^{(x-z/\alpha)^2}dx\right\}. \quad (3)$$

A first trial range was obtained by extrapolating the linear section of a curve drawn through the experimental points. This range was found to be 3.64 cm. Near the tail of the curve, only particles originating near the surface of the foil can be counted. Hence in this region, $z \ll |\alpha x|$, and the curve has very nearly the same shape as it would have with a thin source. We may therefore choose α in Eq. (2) to obtain a good fit to the tail of the curve. For the data shown, $\alpha=0.15$ cm. The dashed curve shows N_y, as calculated from Eq. (2) with these constants. Extrapolation of the tangent of this curve at the point of inflection yields the extrapolated range of (3.81 $\pm$0.03) cm.

Since Eq. (3) cannot be integrated in closed form, the transition from a thin to a thick source was accomplished by numerical integration. It was found that the summation of 20 thin sources, each characterized by a counting rate of 2.3/sec and by the dashed curve of Fig. 2, with an air equivalent distance of 0.125 cm between successive sources, gives the solid curve of the figure, in good agreement with the experimental points.

This apparatus has made possible the attainment of results of fair precision in undergraduate laboratories. The experiment serves as an introduction to counting techniques and to the study of statistical fluctuations. In addition, the analysis of the data is sufficiently simple to be quite clear to undergraduate students, and yet is sufficiently complex to indicate that significant results are not necessarily revealed directly by experimental measurements without some thought as to the interpretation of such measurements.

A number of improvements could undoubtedly be made in the apparatus without undue trouble. A larger area covered with phosphor would increase the counting rate without appreciable sacrifice of favorable geometry. A still greater gain could probably be made if two photomultipliers were used with a simple coincidence circuit[4] to reduce the background count. Finally, the use of 5819 or similar tubes in place of the 931-A type would increase the efficiency and might well eliminate the necessity of cooling, even with weak sources.

[4] W. C. Elmore, Rev. Sci. Instr. 22, 649 (1950).

Summer Meeting of AAPT

The Summer Meeting of the American Association of Physics Teachers will be held in Pittsburgh, Pennsylvania, on June 25, 26, and 27, 1953. Arrangements have been made with the Webster Hall Hotel for room reservations for our members. It is important and necessary that members write to the hotel individually to receive confirmation of their reservations, mentioning their connection with the Association.

Most of the meetings will be held in the air-conditioned rooms of the nearby Mellon Institute.

Titles and abstracts of papers to be contributed to the program should reach the program chairman, Professor Marsh W. White, Department of Physics, Pennsylvania State College, State College, Pennsylvania, not later than April 30. Several symposiums are planned, and active local preparations are being made by the Western Pennsylvania, Central Pennsylvania, and Appalachian Sections in collaboration with Professor Chas. Williamson, Carnegie Institute of Technology.

Relativistic Rocket Theory

W. L. Bade
University of Nebraska, Lincoln, Nebraska
(Received August 14, 1952)

In the relativistic theory of one-dimensional rocket motion in empty space, Ackeret has found the differential equation of motion and the mass-ratio formula. One can obtain a formal solution for the *integrated equation of motion* $X=X(T)$ in the general case by determining X and T parametrically as functions of the proper time s of the rocket.

THE subject of the following discussion is a rocket[1] which moves in field-free empty space. In case either the velocity of the exhaust or that of the rocket itself becomes very large, it is necessary to use relativistic in place of classical dynamics in order to secure exact results. Although such velocities are far out of current technical reach, the problem considered here is of interest both as a legitimate part of the theory of rockets and as an exercise in relativity theory.

This problem may be stated as follows. *A rocket of specified construction moves in field-free empty space, to determine the expected performance.*

The term "specified construction" implies knowledge at least of

1. the initial rest mass m_1 of the rocket;
2. the exhaust velocity $-w$, constant relative to the rocket;
3. the rate defined by $\rho(s)=dm/ds$ at which the rest mass of the rocket changes, as measured by a clock carried by the rocket; s is the proper time of the rocket.

This rate $\rho(s)$ may be any sectionally-continuous nonpositive function of s such that $\int_0^\infty \rho(s)ds+m_1>0$ ($m=m_1$ when $s=0$). The rest mass-ratio, given by the equation $\nu=m_1/m_2=$(initial rest mass)/(final rest mass), can be computed from items 1 and 3 above. The term "performance" means

(*a*) the terminal velocity U_2 of the rocket (the initial velocity U_1 is assumed given);
(*b*) the integrated equation of motion $X=X(T)$ of the rocket in an inertial system.

[1] That is, any sort of reaction-propelled vehicle.

Ackeret[2] has developed the essentials of the relativistic theory of rocket motion in empty space.[3] He has given the fundamental differential equation [Eq. (6) below] of such motion, and has integrated this equation to obtain a formula, Eq. (8), which suffices to determine the final velocity U_2 in terms of the known mass-ratio m_1/m_2. However, he has not discussed the problem of determining the integrated equation of motion. This latter has been solved in the special case of "hyperbolic motion"[4] by Shepherd.[5] A general solution is given here.

To obtain Ackeret's relativistic differential equation of rocket motion, let m be the rest mass of the rocket at any point in the motion, U the velocity of the rocket in an inertial frame $K(XYZT)$, dm' the rest mass of an element of exhaust, U' the velocity of this element of exhaust in K, and $-w$ the exhaust velocity relative to the rocket, a constant.

The rocket is assumed to move along a rectilinear path in the inertial frame K; let the X axis of K lie along this path. Assume that all the elements of exhaust are ejected straight back along the X axis, so that the internal efficiency of the rocket drive is 100 percent and the problem is essentially one-dimensional.[6]

On account of the assumption of "field-free empty space," the rocket, together with its exhaust, constitutes an isolated system. Hence the total four-momentum of this system is constant;

[2] J. Ackeret, Helv. Phys. Acta **19**, 103 (1946).
[3] For a readily available review of Ackeret's work in English, see Seifert, Mills, and Summerfield, Am. J. Phys. **15**, 255 (1947), pp. 267–269.
[4] That is, motion in which the acceleration of the rocket as measured in the instantaneous proper system is constant. See C. Mauguin, Compt. rend. **234**, 1004 (1952).
[5] L. R. Shepherd, J. Brit. Interplan. Soc. **11**, 149 (1952), Appendix.
[6] Shepherd has investigated a case where this assumption is not satisfied. See L. R. Shepherd, reference 5.

that is,

$$\Delta \sum m dx^{\mu}/ds = 0. \tag{1}$$

The summation is over the particles of the system, m is the rest mass of a sample particle and dx^{μ}/ds a component of its four-velocity, while Δ denotes taking the difference of values before and after an interaction among the particles. The relativistic formula for addition of velocities gives in this case

$$U' = \frac{U-w}{1-(Uw/c^2)}. \tag{2}$$

When Eq. (1) is applied to the expulsion of an element of exhaust from the rocket, there results

$$\begin{aligned} d[mdX/ds] + (dX/ds)'dm' = 0, \\ d[mdT/ds] + (dT/ds)'dm' = 0, \end{aligned} \tag{3}$$

in which $(dX/ds)'$ and $(dT/ds)'$ are components of the four-velocity of the ejected material. If now dm' is eliminated from Eq. (3), there results

$$d[mdX/ds] - U'd[mdT/ds] = 0,$$

where

$$U' = (dX/dT)' = (dX/ds)'/(dT/ds)'.$$

Let

$$ds^2 = dT^2 - c^{-2}(dX^2 + dY^2 + dZ^2). \tag{4}$$

Then, since $dY = dZ = 0$ here,

$$\begin{aligned} dT/ds &= [1-(U/c)^2]^{-\frac{1}{2}}, \\ dX/ds &= U[1-(U/c)^2]^{-\frac{1}{2}}. \end{aligned} \tag{5}$$

Making use of Eq. (2) and Eq. (5), one obtains, after algebraic reduction, Ackeret's equation

$$w[1-(U/c)^2]dm + mdU = 0. \tag{6}$$

One readily finds as a first integral the mass-ratio formula

$$\nu = \frac{m_1}{m_2} = \left[\frac{c+U_2}{c-U_2}\right]^{c/2w} \left[\frac{c-U_1}{c+U_1}\right]^{c/2w}, \tag{7}$$

in which U_1 is the initial, U_2 the final velocity of the rocket in K. When U_1 is set equal to zero and Eq. (7) is solved for U_2, one finds, with Ackeret,

$$U_2 = c\left[\frac{\nu^{2w/c}-1}{\nu^{2w/c}+1}\right]. \tag{8}$$

Formulas (7) and (8), determine part (a) of the "performance," since w and $\nu = m_1/m_2$ are known.

When it comes to determining the second part of the "performance," i.e., the integrated equation of motion, a slight difficulty appears. If the function $P(T) = dm/dT$ were given (T = time read by a clock at rest in K), it would be possible to integrate Eq. (6) directly for an equation of motion. However, the rate at which rest mass is lost by the rocket is determined by mechanisms or agencies on board the rocket. Hence, not $P(T) = dm/dT$ but $\rho(s) = dm/ds$ is specified by the construction of the rocket. Because of the Einstein time dilation, these two rates are not the same.

One can overcome this difficulty and obtain the integrated equation of motion (in parametric form) as follows. From Eq. (4), with $dY = dZ = 0$, it is clear that a function $\theta(s)$ exists, such that

$$\cosh\theta = dT/ds, \quad c\sinh\theta = dX/ds. \tag{9}$$

Then

$$U = dX/dT = c\tanh\theta. \tag{10}$$

Substituting Eq. (10) into Eq. (6), one obtains

$$d\theta/ds = -(w/mc)\rho(s).$$

As the quantities on the right are specified by the construction of the rocket, it is possible to integrate and obtain

$$\theta(s) = -\frac{w}{c}\int_0^s \frac{\rho(s)}{m} ds, \tag{11}$$

where

$$m = m_1 + \int_0^s \rho(s) ds.$$

Then, from Eq. (9),

$$T = \int_0^s \cosh\theta\, ds, \quad X = c\int_0^s \sinh\theta\, ds. \tag{12}$$

Equations (12) determine X and T parametrically as functions of the proper time s of the rocket. It is assumed that the system K has been so chosen, and the clocks in K and on the rocket have been so set, that when $s=0$, $X = T = U = 0$.

It is of some interest to note another way of obtaining this result, Eq. (12). The preceding

calculation was carried out entirely in the inertial system of reference $K(XYZT)$. One might instead choose to employ the system $k(xyzt)$ in which the rocket is always at rest at the origin. As this system k is not an inertial one, it is then necessary to borrow techniques from the general theory of relativity.

The problem of accelerated frames of reference has been discussed by Møller.[7] He has shown that the transformation equations from k to K are

$$X = x\cosh\theta + c\int_0^t \sinh\theta\, dt$$
$$T = (x/c)\sinh\theta + \int_0^t \cosh\theta\, dt, \tag{13}$$

and that the metric in k is

$$ds^2 = (1+gxc^{-2})^2 dt^2 - c^{-2}(dx^2+dy^2+dz^2), \tag{14}$$

[7] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd., Bind XX, Nr. 19 (1943).

where

$$\theta(t) = c^{-1}\int_0^t g(t)dt,$$

and $g(t)$ is the acceleration of the origin of k as measured in the instantaneous proper system. From Eq. (13) it follows that the velocity in K of a point fixed in k is

$$U = c\tanh\theta. \tag{15}$$

According to Eq. (14), for a point fixed at the origin of k the coordinate time t is equal to the proper time s. Since the rocket resides permanently at $x=y=z=0$, it follows that Eq. (15) is identical with Eq. (10), $\theta(t)=\theta(s)$ is again determined by Eq. (11), and Eqs. (13) reduce to Eqs. (12).

According to the viewpoint of the system of reference k, the rocket is situated in a gravitational field of intensity $g(t)$ and keeps itself stationary at the origin by use of its reaction engine.

"since bodies, having but one common matter, can be differenced but by accidents, which seem all of them to be the effects and consequents of local motion, I see not why it should be absurd to think, that (at least among inanimate bodies) by the intervention of some very small addition or subtraction of matter, (which yet in most cases will scarce be needed,) and of an orderly series of alterations, disposing by degrees the matter to be transmuted, almost of any thing, may at length be made any thing: as, though out of a wedge of gold one cannot immediately make a ring, yet by either wire-drawing that wedge by degrees, or by melting it, and casting a little of it into a mould, that thing may be easily effected. And so though water cannot immediately be transmuted into oil and much less into fire; yet if you nourish certain plants with water alone, (as I have done,) till they have assimilated a great quantity of water into their own nature, you may, by committing this transmuted water . . . to distillation in convenient glasses, obtain, besides other things, a true oil, and a black combustible coal, (and consequently fire;) both of which may be so copious, as to leave no just cause to suspect, that they could be any thing near afforded by any little spirituous parts, which may be presumed to have been communicated by that part of the vegetable, that is first put into the water, to that far greater part of it, which was committed to distillation".—Quoted by T. S. Kuhn from *The Works of the Honourable Robert Boyle*, Volume 2, Page 375 (London, A. Millar, 1744).

NOTES AND DISCUSSION

Effect of Absorption by the Material of the Prism on Its Resolving Power

MAHENDRA SINGH SODHA
Allahabad University, Allahabad, India

SPARROW[1] has discussed the effect of absorption by the material of the prism on its resolving power on the basis of his criterion. As his criterion has been abandoned in favor of the Rayleigh criterion, expressed in terms of intensity as

$$I_{\min}/I_{\max}=8/\pi^2,$$

the author proposes to discuss the problem on the Rayleigh criterion.

Considering the incidence of a plane wave front on the prism, Sparrow's expression for the intensity of the emergent wave front, in a direction inclined at an angle θ to that of the central maximum, can be put as

$$I=I_0\frac{(1-m)^2+4m\sin^2\alpha}{(\log m)^2+4\alpha^2}, \tag{1}$$

where $m=e^{-kt'/2}$, t' being the longest path traveled by any ray in the prism, k the intensity absorption coefficient and $\alpha=\pi a\sin\theta/\lambda$, a being the width of the emergent wave front.

If the central maxima of two spectral lines (given by $\alpha=0$) have an angular separation $\delta\theta$ corresponding to $\Delta\alpha$, the maximum and minimum of the resultant intensity pattern are given by

$$\begin{aligned}I_{\max}&=I_{\alpha=0}+I_{\alpha=\Delta\alpha}\\&=I_0\left\{\frac{(1-m)^2}{(\log m)^2}+\frac{(1-m)^2+4m\sin^2(\Delta\alpha)}{(\log m)^2+4(\Delta\alpha)^2}\right\}\end{aligned}$$

and

$$I_{\min}=2I_{\alpha=\Delta\alpha/2}=2I_0\frac{(1-m)^2+4m\sin^2(\Delta\alpha/2)^2}{(\log m)^2+(\Delta\alpha)^2}.$$

Rayleigh's criterion for resolution requires

$$\frac{8}{\pi^2}I_0\left\{\frac{(1-m)^2}{(\log m)^2}+\frac{(1-m)^2+4m\sin^2\Delta\alpha}{(\log m)^2+4(\Delta\alpha)^2}\right\}=2I_0\frac{(1-m)^2+4m\sin^2(\Delta\alpha/2)}{(\log m)^2+(\Delta\alpha)^2}. \tag{2}$$

This equation gives one negative value of $(\Delta\alpha)^2$. Taking the square root of the positive value we have, in general,

$$\Delta\alpha=\pi/f(m), \tag{3}$$

where $f(m)$ is a function of m. We have

$$\begin{aligned}\pi/f(m)=\Delta\alpha&=\pi a/\lambda\{\sin(\theta+\delta\theta)-\sin\theta\}\\&=\pi a/\lambda(\theta+\delta\theta-\theta)=\pi a\delta\theta/\lambda,\end{aligned}$$

since θ is small. This result may be expressed in terms of resolving power R by the equation

$$R=\lambda/d\lambda=f(m)ad\theta/d\lambda=R_0f(m). \tag{4}$$

From Eq. (2) we find that the maximum resolving power occurs when $k=0$, $m=1$, and $f(m)=1$, i.e., when there is no absorption. We also find that the resolving power decreases slowly with decreasing m.

An approximate method for the calculation of $f(m)$ for values of m that are of practical importance (0.8 or more) has been developed by the author.

When $m\geqslant 0.8$, $(\log m)^2$ is very small compared to $(\Delta\alpha)^2$ and may be neglected. Moreover, since $\Delta\alpha$ is only slightly different from π we can put $\sin\Delta\alpha=0$ and $\sin\Delta\alpha/2=1$. Introducing these approximations in Eq. (2) we get

$$\frac{\pi^2}{4}\frac{(1+m)^2}{(\Delta\alpha)^2}=\frac{(1-m)^2}{(\log m)^2}+\frac{(1-m)^2}{4(\Delta\alpha)^2}$$

or

$$\frac{\pi^2(1+m)^2-(1-m)^2}{4(\Delta\alpha)^2}=\frac{(1-m)^2}{(\log m)^2}.$$

Putting $\Delta\alpha=\pi/f(m)$ we get

$$\begin{aligned}f(m)&=\frac{2(1-m)}{(|\log m|)}\cdot\frac{\pi}{(\pi^2(1+m)^2-(1-m)^2)^{\frac{1}{2}}}\\&=\frac{2(1-m)}{(1+m)(|\log m|)}\cdot\frac{1}{(1-(1-m)^2/\pi^2(1+m)^2)^{\frac{1}{2}}}\\&=\frac{2(1-m)}{(1+m)(|\log m|)}\left\{1+\frac{(1-m)^2}{2\pi^2(1+m)^2}\right\},\end{aligned} \tag{5}$$

where $|\log m|$ denotes the numerical value of $\log m$.

The author thanks Dr. K. Majumdar and Mr. Y. P. Varshni for their interest in the investigation.

[1] C. M. Sparrow, Astrophys. J. 44, 76 (1916).

Determination of the Coefficient of Surface Tension by the Bubble-Length Method*

JOHN C. SLONCZEWSKI† AND RALPH HELLER
Worcester Polytechnic Institute, Worcester 2, Massachusetts

WHEN a comparatively large air bubble is entrapped under a horizontal plate within the test liquid, an approximate value of the coefficient of surface tension, γ, of the liquid may be obtained by measuring its density ρ and the distance h between the horizontal planes through the vertex of the bubble and its widest (equatorial) section, for then[1]

$$2\gamma/\rho g=h^2 \tag{1}$$

nearly. The left-hand member of Eq. (1) is termed the capillary constant of the liquid.

In an interesting modification of this method of "sessile bubbles," Euverard and Hurley[2] measured the over-all length l of an air bubble of given volume V in a horizontal tube of known internal diameter $2r$, filled with the test liquid. It is true that the mathematical relation which gives γ in terms of ρ, l, V, $2r$ (and possibly the angle of contact) is not known, but Euverard and Hurley were able to show that a smooth curve, approximated satisfactorily by one branch of an equilateral hyperbola, was obtained when the value of $l\rho^{-\frac{1}{2}}$ for each of 18 liquids was plotted against γ (which, like ρ, was taken from tables). Stated mathematically,

$$(\gamma-a)(l\rho^{-\frac{1}{2}}-b)=c, \tag{2}$$

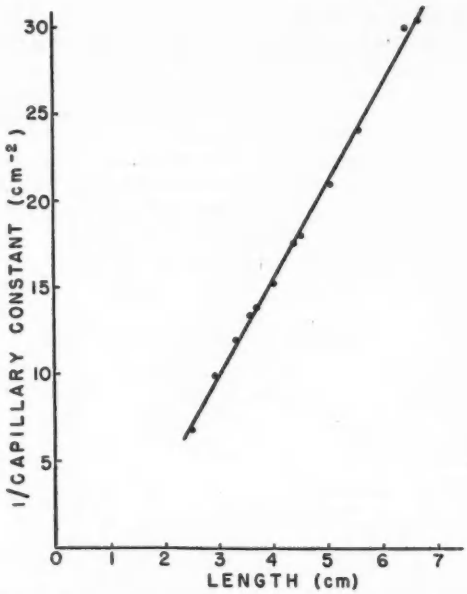


FIG. 1. The reciprocal of the capillary constant, i.e., $\rho g/2\gamma$, as a function of the over-all length, l, of the bubble. ($2r=1.283$ cm; $V=1.00$ cm^3.) The experimental points represent the following liquids, from left to right: Water, glycerol, aniline, benzyl alcohol, nitrobenzene, benzene, methyl alcohol, n-hexane, ethyl ether, D-C 200 silicon, carbon tetrachloride, ethyl bromide.

where a, b, and c are constants determined experimentally for given V and $2r$. With V and $2r$ fixed, it was, therefore, possible to make a fairly accurate estimate of the value of the coefficient γ of a test substance by merely measuring the horizontal length, l, of the air bubble with vernier calipers, which can be done far more easily and more quickly than the measurement of a small[3] vertical dimension such as h.

We have repeated the experiment of Euverard and Hurley, using 12 liquids (mostly organic and chemically pure) at room temperature, and our observations fit a hyperbolic curve of the form of Eq. (2) fairly well. It was possible to hold the volume, V, of the bubble (of the order of 1 cm^3) to within $\frac{1}{2}$ percent of a given value for all liquids by employing the following procedure. A graduated pipette was connected to one end of the tube, the tube was filled completely with the liquid, a short capillary tube was attached to the other end, and the desired volume of air was then admitted through the capillary tube by drawing an equal volume of liquid into the pipette. The length l of the bubble was usually measured by us with the aid of a small micrometer microscope.

As to the mathematical relation among the various quantities involved, the variables in Eq. (2), with V and $2r$ held constant, are γ, ρ, and l. Dimensional reasoning suggests that the "capillary constant," $2\gamma/\rho g$, should be a function of the measured length l (with coefficients depending on V and $2r$), for $2\gamma/\rho g$ is of the dimension of the square of a length (as may be seen from Eq. (1)). Hence $2\gamma/\rho g=f(l)$. When the reciprocal, $\rho g/2\gamma$, of the capillary constant was plotted as a function of the length, l, of the bubble, a practically straight line of the form

$$\rho g/2\gamma=Al+B \tag{3}$$

was obtained. Our data, secured with a polarimeter tube of 1.283 cm internal diameter and a bubble of 1.00 cm^3 volume, are shown in Fig. 1. The straight line found has the slope $A=5.0$ cm^{-3} and the intercept $B=-6.1$ cm^{-2}. The data of Euverard and Hurley likewise fall satisfactorily on a straight line of the form Eq. (3) when so plotted. Using their observations (with $2r=0.8$ cm and $V=0.5$ cm^3) we find $A=9.5$ cm^{-3} and $B=-10.4$ cm^{-2}. Once this straight line has been determined for a tube of given diameter and an air bubble of definite volume, the coefficient γ of a test liquid (of known density) can, therefore, be estimated with fair accuracy from the measured length of the bubble.

Additional investigation of the bubble-length method, carried out both theoretically and experimentally, should prove worth while.

* A portion of the thesis submitted to the Department of Physics, Worcester Polytechnic Institute, by John C. Slonczewski in partial fulfillment of the requirements for the degree of Bachelor of Science (June, 1950).

† Now National Science Foundation Fellow, Department of Physics, Rutgers University, New Brunswick, New Jersey.

[1] See, for instance, B. L. Worsnop and H. T. Flint, *Advanced Practical Physics for Students* (Methuen and Company, London, 1951), ninth edition, pp. 119–122. The experimental method is due to Georg Quincke, Pogg. Ann. 139, 7–8, 11–14 (1870).

[2] M. R. Euverard and D. R. Hurley, Anal. Chem. 21, 1177–1180 (1949). We wish to thank Professor K. L. Mayer and Professor E. D. Wilson for bringing this work to our attention.

[3] At room temperature, h as given by Eq. (1) ranges from 0.18 cm for ethyl bromide to 0.39 cm for water.

Equivalent Networks of Electron Tubes

SAUL ARONOW
Harvard University, Cambridge, Massachusetts

THE four-terminal equivalent network of an electron tube is well known. However, the student is often confused by the variety and arbitrariness of the sign conventions used by different authors. Clarity in understanding equivalent networks was not very important when only the simplest methods of tube operation were in vogue. But now that the unconventional is ordinary, with new types of "tubes," such as transistors, etc., and new techniques of circuitry, it becomes imperative that uniform, easily remembered conventions be adopted.

The reason for the confusion is a conflict of intuition and mathematical necessity. Consider an active four-terminal network, *arbitrarily* assuming directions for the terminal currents and voltages as in Fig. 1. There is a general functional relation

$$i_p=f(v_p, v_g), \tag{1}$$

FIG. 1. Four-terminal "black box" for triode.

which leads to an identity in the partial derivatives,

$$\left(\frac{\partial v_p}{\partial i_p}\right)_{v_g}\left(\frac{\partial i_p}{\partial v_g}\right)_{v_p}=-\left(\frac{\partial v_p}{\partial v_g}\right)_{i_p}. \tag{2}$$

In setting up the small signal, linear, equivalent circuit for the system, we would like to consider the three derivatives as physically significant coefficients, and as such to have positive numerical values. But the mathematics requires that at least one of them be a negative number. Early circuit designers felt that it was "natural" for an amplification to be a positive quantity. Hence, they defined the amplification factor as the negative of the derivative,

$$+\mu_{gp}=-\left(\frac{\partial v_p}{\partial v_g}\right)_{i_p},$$

since the static characteristic, in the negative grid region, has a negative slope. This convention leads to the common equivalent circuit having a distant-controlled generator with inverted polarity.

If the tube draws grid current, we may set up an identity for the grid similar to Eq. (2):

$$\left(\frac{\partial v_g}{\partial i_g}\right)_{v_p}\left(\frac{\partial i_g}{\partial v_p}\right)_{v_g}=-\left(\frac{\partial v_g}{\partial v_p}\right)_{i_g}. \tag{3}$$

Now defining

$$\mu_{pg}=+\left(\frac{\partial v_g}{\partial v_p}\right)_{i_g}, \quad \text{and} \quad g_{pg}=+\left(\frac{\partial i_g}{\partial v_p}\right)_{v_g},$$

μ_{pg} appears, from the slope, as a positive and g_{pg} as a negative number. Again, if the tube is operated in the dynatron region of its static characteristics,[1] the slopes of the characteristics change and all three coefficients may be negative. To redefine the signs of the coefficients as the operating region changes would be hopelessly involved.

Apparently in recognition of these difficulties, the Institute of Radio Engineers Standards[2,3] have been revised to establish a logical, uniform procedure. The Standards define each coefficient as the positive partial derivative, i.e., with numerical value equal in magnitude and *sign* to the slope of the appropriate characteristic curve. The Standards specify the polarities of the equivalent network in terms of current generators as in Fig. 2(a), corresponding to the linear equations

$$I_1=y_{11}V_1+y_{12}V_2,$$
$$I_2=y_{21}V_1+y_{22}V_2.$$

This may be converted to voltage generators as in Fig. 2(b),

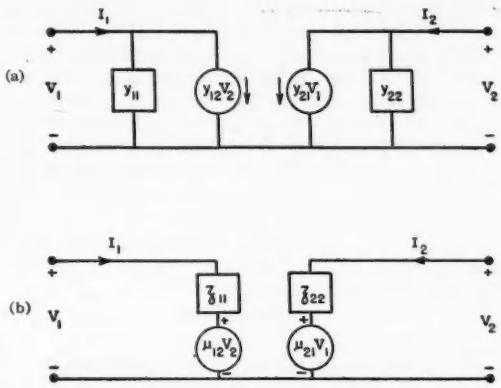


FIG. 2. Linear equivalent networks for electron tubes. (a) Current generators. (b) Voltage generators.

corresponding to

$$V_1=z_{11}I_1+\mu_{12}V_2,$$
$$V_2=\mu_{21}V_1+z_{22}I_2.$$

It should be noted that the coefficients are defined in a very flexible manner, with currents and voltages measured at whatever points it is convenient to designate as external terminals, so that with proper interpretation, the network may be used for many devices. The importance of a systematic procedure becomes more evident when it is noted that while the Standards, as always, have fallen behind recent technical developments, the basic principles still apply. The networks in Fig. 2 exhibit only 8 of the 24 possible partial derivatives. In transistor circuits, for example, it is more convenient to define current amplification factors or to use systems where one current and one voltage are considered dependent variables. However, if the basic definitions and sign conventions of the Standards are followed, any arbitrary selection of variables may be made and the resulting equivalent network written down without hesitation.

[1] E. L. Chaffee, *Theory of Thermionic Vacuum Tubes* (McGraw-Hill Book Company, Inc., New York, 1933), p. 153.
[2] IRE Standards on Electron tubes; Definitions of Terms, 50 IRE 7.S1, Proc. Inst. Radio Engrs. 38, 426 (1950).
[3] IRE Standards on Electron Tubes, 50 IRE 7.S2, Proc. Inst. Radio Engrs. 38, 935 (1950).

AMERICAN JOURNAL OF PHYSICS VOLUME 21, NUMBER 4 APRIL, 1953

LETTERS TO THE EDITOR

An Extension of the Conical Pendulum Problem and Its Demonstration

IT is a simple exercise to show that the height h of the conical surface swept out by the string and bob in a conical pendulum motion is given by g/ω^2. That is, h depends *only* on the speed of rotation and not upon the weight of the bob or the length of the supporting string. If, then, a number of *different* masses, suspended from the same point by strings of *different* lengths, are given the same angular velocity about a vertical axis through the point of suspension, should not the masses all lie in the same horizontal plane? This is a necessary deduction, but not intuitively clear. Indeed, this is exactly the kind of situation I

have referred to earlier where, as I said, I believe the result but would like to see it.

The demonstration is easily accomplished by suspending bobs of different masses on strings of different lengths from a shaft held in the chuck of a rotator. A short piece of rod or metal tubing drilled along several diameters makes a good shaft. The method of fixing the string to the shaft is critical but not difficult to adjust. With different colored bobs the demonstration has a measure of beauty and lends itself nicely to shadow projection.

JULIUS SUMNER MILLER

Ford Foundation Fellow
10303 Mississippi Avenue
West Los Angeles 25, California

The Ultrasonic Interferometer as a Screw Calibrating Instrument

PRECISION screw calibration requires a small standard of length, precisely known, which can be easily compared with the linear advance of the screw being calibrated. The best standard of length is the wavelength of light, so that the optical interferometer may be considered the ultimate screw calibrator despite the fact that the tediousness of the screw-to-standard comparison makes its use rather impracticable. But recent improvements in the acoustic interferometer, which uses the wavelength of sound as the standard of length, suggest that this instrument might be more successful in calibration applications where the ultimate in accuracy is not required.

The principal advantage of the acoustic interferometer as compared with its optical counterpart is its tremendous superiority in speed and ease of the screw-to-standard comparison. The ultrasonic interferometer developed by Barthel and Nolle[1] for measurement of dispersion of ultrasonic sound in liquids is capable, with some modification, of turning out a permanent record of a complete screw calibration within a minute. Preliminary experiments show that the instrument is capable of recording cumulative linear increments of 0.003 ± 0.00004 cm. For a screw with a one millimeter pitch, for example, the size of the increment thus corresponds to a record point about every 11° of rotation of the screw. It seems likely that developmental research could still further reduce the size of the increment and increase the precision.

The instrument referred to above is an interferometer of the two-crystal type in which the separation of the crystals, which are immersed in a liquid medium, is continuously varied. The recording process is equivalent to placing a mark on a rotating drum whenever the receiving crystal has moved a distance of one wavelength of sound in the liquid. This high-precision instrument utilizes low intensity traveling waves, rather than the high intensity standing waves used in most other interferometers, and thereby avoids the errors arising from local heating of the liquid. It surpasses other currently used interferometers in its rapidity of operation as an automatically recording instrument. For further details reference should be made to the original article describing the instrument. Much of the mechanical complexity of the original instrument would be unnecessary in an instrument specifically designed for screw calibration.

ROMARD BARTHEL

St. Edward's University
Austin, Texas

[1] R. Barthel and A. W. Nolle, J. Acoust. Soc. Am. **24**, 8 (1952).

Concerning the Electric Charge on a Moving Vehicle

IT is common knowledge that moving vehicles acquire an electrostatic charge, and the mechanism appears to be frictional. Gasoline trucks are required by law to provide a ground, which may be a dragging chain or strap, and some passenger cars are so equipped. At a ferry dock near here the approaching roadway carries an upright flexible steel strap which is intended to carry to ground any charge which an approaching vehicle may bear. The question as to how great a charge may accumulate is difficult to answer; it is not easily computed theoretically and practical measurements offer trouble. This note is intended to report an observation which points up further difficulties.

On a dry summer day, not far from desert terrain, while driving at a distinctly slow rate, a friend overtook me in his car and came abreast of me on my left. His wife in the front seat reached out and handed me a package of gum. I reached for it with my left arm extended (a dangerous business, admittedly!), but came no closer than three inches, perhaps, if indeed as close as that. At this instant a terrific discharge took place which possessed the classical physiological effects. The shock was momentarily disabling, as a three-inch spark in air can well be. Both of us recovered promptly but I have since wondered what the mechanism of storage of charge might be. Simple frictional effects can produce a substantial charge but leakage from the many sharp edges of an automobile body must inevitably arise, unless some capacitance effect exists. What can we say about this?

JULIUS SUMNER MILLER

Ford Foundation Fellow
University of California
Los Angeles, California

Demonstration of Subjective Harmonics

SUBJECTIVE harmonics produced by nonlinear response of the human ear have been known for a long time; yet they are not easily demonstrated under unimpeachable conditions. What is needed is a high intensity sinusoidal stimulus; and that is not likely to be furnished by any ordinary combination of oscillator, amplifier, and loud-speaker or headphones.

Some years ago there was described in this Journal a thermionic tube circuit with which tuning forks can be driven to large and controllable amplitudes.[1] Tests with a high quality microphone and associated equipment do not show significant distortion in the air-borne sound. Yet, if an observer closely approaches one of our large 128-cycle forks when it is thus driven, he hears what is

known in music as a common chord, with frequencies easily identified as 512, 640, and 768 cycles/sec. (With a little attention a few of the lower and higher harmonics may also be heard.) This effect promptly disappears when the observer moves away from the fork.

The effect is so conspicuous that the observer notices it immediately; he does not need to be told in advance what to listen for. It cannot be due to overtones produced by the fork itself, for these are nonharmonic. The frequencies of the first two overtones of this fork have been measured[2] and were found to be 807 and 2154 cycles/sec.

It seems reasonable to conclude that true subjective harmonics are being observed.

CHARLES WILLIAMSON

Carnegie Institute of Technology
Pittsburgh, Pennsylvania

[1] C. Williamson and W. J. Eisenbeis, Am. J. Phys. (Am. Physics Teacher) 4, 91 (1936).
[2] This measurement is conveniently and precisely made with our apparatus by exciting the driving amplifier with the output of an audiofrequency oscillator and checking the frequencies of the audible response peaks with a Stroboconn.

Trigonometry before Physics?

PHYSICISTS today are offered many and varied opportunities in both the theoretical and applied fields, but strange as this may seem, there are insufficient qualified personnel to fill these positions. Therefore the question arises "why?" A possible answer may be found in the high mortality rate among physics majors in the colleges. Again the question, "why?" A possible reason is that on application to college many find they have inadequate background for physics. For example, today some are applying with little or no knowledge of trigonometry. The college offering the student a choice of taking general physics either in the freshman or sophomore year gives him an opportunity to remedy this deficiency. But to the college offering general physics in the freshman year only, it presents a serious problem.

In an attempt to ascertain the attitude of colleges throughout the country and thus seek a solution to this problem, a questionnaire was circulated in May 1952 to 145 institutions of higher learning that were representative of all important types found in the United States. Its primary purpose was to learn the attitude of other schools toward the question, "Do you regard trigonometry as a requisite or prerequisite for general college physics?" (Incidentally, a question involving trigonometry for the biology major was also inserted.) Most colleges solve that problem by giving general physics in the sophomore year, and those giving it in the freshman year either teach trigonometry in course or run a class concurrently.

The concern, however, is with the physics major and his ability to survive his course. With inadequate background he is always behind, for while new material is presented he does not have the necessary tools to handle it and, therefore, a high mortality rate exists. This is true not only in his physics courses; he also finds himself failing in his mathematics courses. Where, then, does the solution to this problem lie? Judging by the requisites as indicated in many college catalogs it seems that colleges themselves are partly responsible for this condition, for it is difficult, if not impossible, to determine just what subjects are required of the high school graduate. This vagueness could very well lead a high school principal to say, as he did, "You won't have to bother with trig, because you won't have much need of it."

The following are the results of the questionnaire. One hundred and forty-five were sent throughout the United States and one hundred and twenty-nine replies were received.

1. Number requiring trigonometry as an entrance requirement for physics majors		16
2. Number requiring trigonometry as an entrance requirement for biology majors		8
3. If the entrant has not had trigonometry do you permit him to start the General Physics course?		
No	97	
Yes	5	
Pre-requisite or concurrently	17	
Impossible to determine	3	
Recommended	3	
Contradictory	4	
4. Number not requiring trigonometry as a prerequisite for physics majors		5
5. Number requiring trigonometry as a prerequisite for physics majors		
Yes	113	
No	5	
Recommended	5	
Taught in Course	3	
Impossible to determine	3	

Note that five colleges (two universities, one Eastern college and two Midwestern state colleges) do not consider trigonometry as a prerequisite, but their remarks indicate otherwise. One reported, "Some introduction to trigonometry is given in the first semester of the first year in physics. In the second semester familiarity with elementary trigonometry is assumed, though no course is required as prerequisite." One wonders how the familiarity is obtained.

Another reported, "Physics majors must take calculus before junior year. Doubtful if any student who has not had math through trig has ever tried to major in physics. It would be very difficult, but not impossible." (!!)

The strangest reason of all was given by a college from the Midwest, "We do not consider it the privilege of the university to dictate to the secondary school that they require or even offer trig."

Still another wrote, "This college has no high school math requirement. All students, however, are required to take a precollege math test."

The above are direct quotes from representative colleges in different sections of the country.

On the positive side of the question, not all are in agreement. One wrote that he could "teach sufficient trig in two or two and one-half hours." Another stated "We think we can teach sufficient trig in course for the student to get by." Still another, "Trig is not an entrance require-

ment in any curriculum. All students in technical curricula are required to spend 75 hours (5 credits) in algebra and trigonometry."

But on the definitely positive side, a Western college wrote, "Mathematics instruction in the public schools of this state has reached the point where relatively few entering students have had trigonometry. We have, therefore, been compelled to introduce this subject into a unified mathematics course for the freshman year which is required of all students taking physics."

Akin to this problem, Dr. Geer of the University of Southern California remarked[1] that "only 18 students out of 1800 complete trigonometry in a high I.Q. high school in Los Angeles." He points out the shortage of scientists and engineers and states that one of the contributing factors is "the omission of this course in many of our high schools, even those preparing a boy to pursue a science course in college." The present survey entirely agrees with the above.

The following points are offered as possible solutions to the problem:

1. Greater unanimity among the professors in the field both on the college level and on the secondary level.
2. Colleges should state definitely, clearly and concisely their general entrance requirements and special requirements for scientific courses. Special provision should be made for the unusually capable applicant who does not have the necessary background.
3. A definite effort to integrate the high school curricula with the specific required college curricula.

The deans of studies in the secondary schools would then know definitely the collegiate requirements and plan the secondary curriculum accordingly. The high school graduate would then be adequately prepared to embark upon his chosen major in college. Hence, he would not be continually "one year behind," nor would he be forced to withdraw for what is weak background, not low mentality.

W. A. Murtaugh, O. P.

Providence College
Providence 8, Rhode Island

[1] Am. J. Phys. **20**, 397-398 (1952).

AMERICAN JOURNAL OF PHYSICS VOLUME 21, NUMBER 4 APRIL, 1953

ANNOUNCEMENTS AND NEWS

Modern Physics. F. W. Van Name, Jr. Pp. 360+vii. Prentice-Hall, Inc., New York, 1952. Price $5.50.

The teaching of an undergraduate modern physics course usually presents something of a problem with regard to organization of material. Unlike mechanics or electricity and magnetism courses a logical, systematic approach is almost impossible, resulting both from the catchall nature of such courses in which everything except classical physics is included, and from the relatively sophisticated viewpoint necessary to unify the aspects of physical phenomena usually treated. Hence the available texts fall naturally into two classes, depending on whether the authors are primarily interested in presenting a great deal of miscellaneous information as palatably as possible or in providing a sound and adequate prelude to further work; i.e., they are either topical discussions or introductions to quantum theory. There is much to be said for each of these approaches, but in general the former is more useful for the less advanced student requiring a broad although not necessarily detailed knowledge of modern physics.

Modern Physics attempts to fulfill this latter function by presenting a well-written survey of the field intended for a full year course. The book is divided into three sections, devoted to The Electron, Atoms and Molecules, and Nuclei and Nuclear Particles, with each of the chapters covering a specific topic. Chapters dealing with kinetic theory, low temperature phenomena such as superconductivity and the behavior of the He II, cosmic rays and mesons, and quantum mechanics supplement the orthodox ones on atomic spectra, x-rays, special relativity, etc. A discussion of nuclear fission and its applications is included. The author makes a distinct effort to encourage understanding by providing complete derivations of the more important equations and by the inclusion of worked-out numerical calculations, in many cases. However, there is a lack of uniformity about the levels of the various chapters, with the one on low temperature physics, for example, being almost entirely qualitative while the quantum mechanics part is rather a difficult morsel for the student to swallow at the stage in his education when he would be using a book of this type. It is likely that most instructors would prefer a more elementary exposition, although there is no reason why this material could not be covered adequately if a healthy amount of time were devoted to it.

At first glance this book seems to consist of a larger ratio of words to formulas than one encounters elsewhere in literature of this kind, but it is necessary to consider that the problems appended to each chapter are an integral part of the text and their solution is necessary for full understanding of the material. Keeping this in mind, *Modern Physics* is welcome addition to the many works already in use, some of which it is certain to replace.

Arthur Beiser
New York University

Physics and Medicine of the Upper Atmosphere. Edited by CLAYTON S. WHITE AND OTIS O. BENSON, JR. Pp. 611+xxiv, Figs. 150, plates 46. University of New Mexico Press, Alburquerque, New Mexico, 1952. Price $10.00.

Rather than a treatise or a text in the usual sense, the book presents an edited group of reports giving the proceedings of a symposium on the field indicated by its title, held in San Antonio, Texas, on November 6, 7, 8, and 9, 1951, and sponsored by The Air University School of Aviation Medicine. The arrangements were made by The Lovelace Foundation for Medical Education and Research of Albuquerque, New Mexico.

Many of the thirty-eight contributors are well known and outstanding physicists or medical men, and the list includes high ranking officers of the United States Air Force, which officially supported the project. The subject matter deals primarily with the physics of the extreme upper atmosphere with emphasis on those aspects which present hazards to aviators who now, or eventually, may operate in this region. The medical contributors to this text have attempted to appraise the seriousness of these hazards from the standpoint of the aviator. The writers do not question the ability of the aeronautical engineer to produce craft capable of reaching and traveling in the aeropause, as the border region between outer space and the atmosphere of the earth is called. They take it for granted that the limitations to such travel are human, not mechanical. Certainly the current developments of rockets and of rocket-type craft of various designs support this point of view. Actually much of the material in the book is based on scientific data secured recently by instruments carried aloft by rocket propulsion. Many important but only lately discovered facts are quite at variance with the notions long held and taught in physics classes, and physicists generally would do well to read carefully this new book before using again any old "lecture notes" dealing with the upper atmosphere. This is particularly true with respect to air temperatures, air composition, and to the nature and the intensity of the various radiations of importance in the aeropause. The present tidal wave of interest in nuclear physics finds added strength in the new observations made in high flying planes and in results obtained through the use of rockets. But the medical contributors are genuinely concerned as to the possible effect of particle radiations, as well of meteoroids and the like, on any human venturing into the space-atmosphere border regions, or even into the highest layers in the atmosphere now reached by manned, rocket-driven planes. The enthusiasm of youth as well as of many uninformed adults for "space travel" would be cooled by reading in this book scientifically reliable reports on the many difficult problems which must be solved before a human may try to reach the altitudes now attained by unmanned rockets.

The subjects stressed in the more than forty papers included deal largely with the problems of temperature, pressure, and composition of the air, and electromagnetic and particle radiations. Only a few chapters are heavily laden with theory. Both the physicists and the medical scientists may find indicated new types of experimental techniques with which they have had no experience in the laboratory. The question as to whether human beings can tolerate the cosmic radiation found to exist at extreme altitudes is discussed by several of the contributors, each expressing his own opinions on the subject. Whether fortunate or otherwise, the contributions on any subject may not fully agree, but this might well be expected on a subject of which the element of speculation is so largely involved. This does suggest what some may criticize—that there is a great deal of overlapping, and disproportionate space may be given to subjects of minor importance. But the reviewer considers this inevitable in a book of collected papers unless the editors should choose to completely revamp much of the material. To have done this would have not only dimmed the individualities of the contributors but greatly delayed the publication of this up-to-the-minute material. The authors, editors, and the publishers are all to be congratulated on having published such a volume in less than a year from the time the material became available. The book presents not only a great deal of new material directly but includes at the ends of the chapters extended bibliographies. These should be extremely valuable to research workers and to all others concerned with the problems discussed. A most interesting feature of the book is the large collection of plates included at the end of the text proper. Without doubt this book will find a place in the libraries of physics departments and of medical schools generally. It is well printed.

E. L. HARRINGTON
Riverside, California

Textbook on Sound. J. W. WINSTANLEY. Pp. 239, Figs. 154, 14×22 cm. Longmans, Green and Company, London, 1952. Price $2.60.

This book is designed, in the English market, for what amounts to high school senior or college freshman level. At this level the author treats the usual gamut of acoustical topics: wave motion, audible acoustic phenomena, ultrasonics, musical and architectural acoustics, instrumentation. The subject treatment is rather uniform in difficulty, and the text would probably be sufficiently challenging for students at these levels. However, since acoustics courses in this country are generally given to third- or fourth-year college students, the book is probably at too low a level to be practicable as a text.

Among the better features of this book are the rather large number of worked problems, as well as a good collection of meaningful problems for homework assignments.

One may also note what appears to be a standard British textbook practice, namely, the description of a large number of interesting experiments which could be performed in the laboratory or at the lecture bench. It is this reviewer's opinion that if some of these experiments are actually performed, the recital can be of value, but that otherwise they tend to appear as so much superfluous material.

In several places, the author has omitted important modern developments. Thus the chapters on ultrasonics do not contain any account of modern pulse techniques, nor of the use of barium titanate elements as transducers. Also, the discussion of sound refraction could well use some mention of the use of plastic lenses. In the discussion on Kundt's tube (p. 151) it would be appropriate to discuss measurements of velocity of compressional waves in rods by applying dust powder directly to the rod, as was done by Röhrich and many subsequent observers.

It is the reviewer's task to point out that there exist a number of incorrect or misleading statements on ultrasonics and wave motion. The statement that CO_2 becomes "opaque" to sound waves at higher frequencies (p. 45) is certainly misleading, while the present upper limit of high frequency sounds is 500 times higher than the value of two megacycles given on p. 35. Finally, few people in underwater sound work will believe that "sound ranging in the sea also has the advantages that the sea is homogeneous and there is no need for a meteorological correction" (p. 50).

Because of statements such as these, it is difficult for the reviewer to recommend this book with much enthusiasm. It was not, we believe, designed for the American market.

ROBERT T. BEYER
Brown University

Comets and Meteor Streams. J. G. PORTER. Pp. 123+xi. Figs. 31, 16×24 cm. John Wiley and Sons, Inc., New York, 1952. Price $5.25.

This small book is No. 2 in the International Astrophysics Series, edited by M. A. Ellison, The Royal Observatory, Edinburgh, and A. C. B. Lovell, Professor of Radio Astronomy, University of Manchester. The editor's note states that the aim of this series "is to provide a collection of authoritative volumes dealing with the main branches of Astrophysics and Radio Astronomy." The reviewer's duty is to appraise the contents of the book on the basis of this clearly expressed intention.

The book from this point of view is a dismal failure. It is nothing but a very short, even if excellently written, monograph on the orbital theory of comets and meteors. Of astrophysics, as this term is generally understood, the author is quite innocent. Spectra of comets are treated in less than a page, spectra of meteors are not even mentioned. The brightness of comets deserves somewhat more attention—two whole pages are devoted to it; the brightness of meteors is again completely overlooked. The internal structure of the comet is simply mentioned in a half-page discussion of the problem of light pressure. One wonders what the author and editors mean by astrophysics.

To any person familiar with the subject it is clear that comets are intimately connected with meteor streams, meteorites, and diffused matter in space. Precise observation of all these phenomena is difficult, owing mostly to the transitory nature of most of these objects. We have, therefore, a large collection of observational data, often mutually exclusive and contradictory. Nevertheless, considerable progress has been made in recent years in the collection of better data and theoretical interpretation of the spectra and luminosity of comets. Practically nothing of this is mentioned in the reviewed book. So far as meteors are concerned, the new and very promising method of radar observation is discussed in considerable detail (pp. 72–74 and 99–103), but the contribution here is again toward the dynamical rather than physical properties of meteors.

The quarrel is with the title and purpose of the book. If the author had called it "Orbits of Comets and Meteor Streams," he might have been congratulated in carrying out his task in so short a space. There are many useful tables in this book such as a list of short period comets observed at more than apparition (p. 37), comet groups (pp. 48–49), cometary meteor radiants (p. 92), etc. The subject is extremely condensed, and it is doubtful that a person without a previous study of the theory of perturbations will derive much of an idea of this involved subject from the few pages devoted to it (pp. 54–65). The last 14 pages of the book are occupied with instructions for the calculation of a comet's ephemeris and of meteor orbits. If we subtract about eight pages of bibliography appended to various chapters, we find that the whole story of comets and meteors is told in about 96 pages. This much space would be necessary to treat fully and coherently the subject of the spectra of comets alone.

There are six very good plates showing comets and meteors entirely unconnected with the text.

The price $5.25 for such a small book seems to be out of all reason. How can a student or even a research worker be expected to invest so much in a book dealing with the restricted aspect of a very restricted problem? On the basis of pages (text, illustrations, formulas, etc., being comparable) the recent book *Astrophysics*, 703 pages (editor J. A. Hynek) published by the McGraw-Hill Co. should cost nearly $30.00. In reality its price is only $12.00.

N. T. BOBROVNIKOFF
Perkins Observatory

A University Text-Book of Physics, Volume III, Heat. 11th Edition. J. H. AWBERY. Pp. 439+x, Figs. 197, 16×23 cm. Charles Griffin & Company, Ltd., London, 1952. Price $5.50.

Many physicists undoubtedly enjoyed reading at least one of the volumes of Poynting and Thomson's well-known *Text-Book of Physics* at some time or other. This was an elementary but exact presentation of physics by distinguished physicists. But textbooks, unlike creative ideas, rarely outlive their authors. This, I am afraid is true of Poynting and Thomson's "Heat," now partly rewritten by Mr. Awbery, and available in a new edition, printed and bound in the best English tradition.

While modernized in many respects, the book is certainly not up-to-date. With the exception of the references in some sections added by Mr. Awbery, most references are to old sources and texts no longer easily available. This is in good order when important original work is referred to, but little purpose is served when several methods of meas-

uring a single quantity are described, some only differing in detail, and every one of them antiquated. Thus, seven methods of measuring the solar constant are outlined, yet none later than 1910. The chapter on water in the atmosphere (a field where progress has been particularly rapid since the advent of aviation, sondes and radio) similarly summarizes the knowledge and experimental equipment of the 1900's and includes the *canard* (dated 1905), now famous among geophysicists, that the masses of individual raindrops in the same rainfall are in the ratio 1:2:4:8.

Considerable space is devoted to thermodynamics. The first law is presented as the result of a number of experiments on the "mechanical equivalent of heat"; altogether too much attention is devoted to a discussion of J, which is logically nothing but a measure of the heat capacity of water. The discussion of the second law is good and includes the usual topics—Clausius' inequality, entropy, and definition of temperature. There is also a chapter on the thermodynamics of radiation.

An excellent, but very elementary, account is given of the kinetic theory of gases, including transport phenomena; this is supplemented at the end of the book by a chapter on statistical mechanics and quantum theory. Even though the more difficult mathematical details are avoided, it is doubtful whether a student who turns to this book for instruction in the elementary topics can benefit much from it. More advanced readers, on the other hand, will more likely than not be put off by the elementary character of the remainder of the book.

This, as some other features, suggest that this book will probably not be found suitable as a text for any *one* course on heat, though it might serve as a companion for the student from his first contact with the subject of heat through to his graduate work. The student would miss, however, a more adequate coverage of recent developments and a treatment of the more philosophical parts of the subject (such as the relation of experimental heat to thermodynamics to kinetic theory). He might also deplore the absence of problems.

WALTER HITSCHFELD
McGill University

AMERICAN JOURNAL OF PHYSICS VOLUME 21, NUMBER 4 APRIL, 1953

RECENT MEETINGS

Illinois Section

The fall meeting of the Section was held at Rockford College, Rockford, Illinois, on Friday and Saturday, November 7 and 8, 1952, with an attendance of approximately 40. A varied program, organized by the local chairman, PROFESSOR A. FRANCES JOHNSON, was presented.

1. Preparation of specimens for use in an electron microscope. MILDRED BULLIET, *Rockford College.*—A demmonstration was made by allowing a drop of a solution of parlodion in amyl acetate to spread into a thin film over the surface of water in a wide dish. Small round screens of fine mesh which fit into the specimen holder of the microscope were placed on the film. The film and screens were then picked up on a cover glass and allowed to dry. A drop of a suspension in water of particulate matter to be viewed in the microscope was placed on each screen. When dry the specimen was ready to be used. A variation of this method was made in showing how the small particles might be dispersed in a parlodion solution before it was dropped on the water.

The preparation of Formvar and silica replicas of surfaces was described. Pictures of particles and replicas made at Rockford College were shown.

2. Two new physics films, circular motion, Carnot cycle and Kelvin scale.

3. Modern applications of an old theory. LEONARD R. INGERSOLL, *Emeritus Professor of Physics, University of Wisconsin.*—While heat-conduction theory is, by and large, a century old, it has many very modern and up-to-date applications. These include thermal insulation in all its various aspects, with special graphical solutions for cases like corners and edges. Subterranean temperatures can be treated by Fourier analysis, and this has been applied to a study of the "thermal history" (postglacial time calculations) of northern Michigan on the basis of special temperature measurements in the deep Calumet and Hecla copper mines.

One of the most interesting and recent applications is in connection with the heat pump. When this rapidly growing method of year-round air conditioning uses a ground coil as its source (or sink) of heat, line-source conduction theory can be used to good advantage, since it is the only practical theoretical way of avoiding the various pitfalls which have so far retarded progress with this phase of heat-pump design.

4. Adding two distributions. EDWARD C. VARNUM, *Barber-Colman Company.*—By adding random pairs selected from two independent distributions, a third distribution is formed having a mean and variance equal, respectively, to the sum of the means and variances of the original distributions. Assuming that the distribution of measuring-instrument errors is independent of the distribution of actual data in a physical test, the above property may be applied to the study of an observed data distribution. When the differences of the random pairs are taken in a specified order, the variances are also additive. This result has been applied to the study of overlapping tolerances of mating parts. A chart for determining the proba-

bility of interference of two overlapping normal distributions was given to each member of the audience, and the use of the chart was illustrated by means of a numerical example expressed in terms of the distributions of shaft and hole diameters in an assembly process.

5. Impressions of great German scientists as recalled by a student of physics in Munich in the early 1930's. MARGARETE BRUCH, *Winnebago.*—The speaker, daughter of the late professor of chemistry, Dr. Richard Willstätter, had the opportunity both at home and as a student at the University in Munich to know Professor Wilhelm Conrad Röntgen, tall and impressive, with a soft voice and tiny handwriting; his successor Wilhelm Wien, known for his work on blackbody radiation; and Wien's successor, Walter Gerlach. She also had the good fortune to study under Arnold Sommerfeld, well known for his explanation of the fine structure of spectra and one of the first to accept Bohr's theory. An inspiring teacher, he developed, in his lectures before his students, his theory of the electron gas obeying Fermi statistics. She heard beautifully clear lectures by Max Planck and challenging lectures by Albert Einstein, whose interest in the most varied problems was shown in his talking with the same verve on unified field theory and on the forces that make moist sand cling together. (This talk was illustrated by portraits and autographed letters.)

6. An operational approach for the classroom development of basic concepts and Newton's laws. CARL J. RIGNEY, *Northern Illinois State Teachers' College.*—In this instructional method, care is taken to employ an explicitly outlined scientific procedure which involves operational definitions only. The procedure of physics is formulated as "the development of our concepts of what is basic in nature by a systematic use of what is basic in our experience with the physical world." Length, time, and force are agreed on as mechanical concepts that are basic in our experience. Weights are used to calibrate a spring scale in arbitrary force units. The findings of an imaginary scientific expedition show that the weight of an object at any location is directly proportional to the free-fall acceleration at that location. The constant of proportionality for a chosen object is selected arbitrarily as the unit mass, and the force unit—no longer arbitrary—follows.

7. What can we do for the less-than-mediocre students now crowding our colleges? CHALMER N. PATTERSON, *Bradley University.*—The seriousness of this problem is widely acknowledged. Quick elimination of such students is rejected because (a) many weak students have, by properly guided hard work, achieved considerable success; (b) even failing students *can* learn much of value to themselves when theories are properly presented with careful illustration from daily life.

Three possible solutions are suggested: (1) Easy informational courses, credited toward graduation at only half value, with D grade, and only after the student has earned the required grade point average in other classes. Objection: Such students are already labeled "Failure." (2) Survey courses with one extra period, open to all but required of weak students, scheduled each week. Only D grade permitted for student required to attend extra period unless he shows notable improvement as semester progresses. (3) Similar extra periods for students entering courses for Arts students with minimum prerequisites.

The writer recognizes that there may be many other ways of trying to improve a bad situation.

Group activities included a demonstration of an electron microscope in operation, a visit to the Air Distribution Development Laboratory of the Barber-Colman Company, a discussion and demonstration of a Philbrick analog computer, and a tour of the laboratories of the Woodward Governor Company. At the business meeting of the Section, PROFESSOR O. L. RAILSBACK, University of Illinois, Navy Pier, Chicago, reported upon recent national meetings, and PROFESSOR GLEN Q. LEFLER, Eastern Illinois State College, presented a report of the Committee on Constitution Revision. A dinner, a luncheon, and a special program for the ladies offered pleasant occasions for social contacts.

O. L. RAILSBACK
A. FRANCES JOHNSON

Symposium on Molecular Structure

The Symposium on Molecular Structure and Spectroscopy will be held at the Department of Physics and Astronomy, The Ohio State University, as usual in June, from June 15 to June 19, 1953. There will be discussions of the interpretation of molecular spectroscopic data as well as methods for obtaining such data. In addition, there will be sessions devoted to those phases of spectroscopy of current interest. A dormitory will be available for those who wish to reside on the campus during the meeting. For further information, or for a copy of the program when it becomes available, write to Professor Robert A. Oetjen, Department of Physics and Astronomy, The Ohio State University, Columbus 10, Ohio.

VAN NOSTRAND FOR COLLEGE TEXTS

Important New Books for Classroom Use

MICROWAVE THEORY AND TECHNIQUES

By H. J. REICH, P. F. ORDUNG, H. L. KRAUSS, and J. G. SKALNIK

The comprehensive treatment of the theory and applications in this new text is carried from field theory through transmission lines, waveguides, radiation, and microwave amplifiers and oscillators. Thus, the textual material plus the generous references to papers and books, make this a comprehensive text and a valuable reference book. Ready in June.

ELECTRONICS
PHYSICAL PRINCIPLES & APPLICATIONS

By A. O. WILLIAMS, Jr.

This new text gives a rounded, physical introduction to the subject, without exhaustive enumeration of special circuits, but with constant attention to the associated phenomena of electricity, magnetism, radiation (including optics), and atomic physics that make electronics an interesting and useful tool to the physicist. Especially suited for physics, chemistry and other majors. 306 pages. $4.50

PRINCIPLES OF MODERN ACOUSTICS

By GEORGE W. SWENSON, Jr.

This new text develops a number of important analytical techniques. It will appeal to the scientist because of its thorough explanation and use of equivalent circuits for mechanical and acoustical systems. It concentrates almost entirely on methods and theory, rather than details of the myriad of modern applications and devices. It is spiced with a number of example applications. Ready in April.

GENERAL COLLEGE PHYSICS

By ERTLE LESLIE HARRINGTON

The outstanding feature of this text is the selection of illustrations and applications. Here the student will find, repeatedly, applications of physics with which he has become acquainted but without recognizing them as such nor even understanding the physical principles involved. Author took great care to see that his text would support the standard laboratory course. 787 pages. $6.75

REVISED 2nd EDITION

MODERN COLLEGE PHYSICS

By HARVEY E. WHITE

THE OUTSTANDING CHANGES IN THIS NEW TEXT are the considerable expansion of alternating current theory and the use of the mks system of units in electricity and magnetism. Chapters on cosmic rays, nuclear physics, and atomic energy have been largely rewritten to bring them up to date. All new problems are given at the end of each chapter. 823 pages. $6.75

EXAMINATION COPIES SENT ON APPROVAL

D. VAN NOSTRAND COMPANY, INC.

PUBLISHERS SINCE 1848

250 FOURTH AVENUE NEW YORK 3, NEW YORK

CONTENTS

AMERICAN JOURNAL OF PHYSICS VOLUME 21, NUMBER 4 APRIL, 1953

Construction of a Diffraction Grating Spectrograph . . . . . R. L. PURBRICK 241

On a Routine Analytic Method for the Solution of Problems in Statics . . . . . A. W. SIMON 244

Constant Battery Wheatstone Bridge . . . . . T. TOWNSEND SMITH 247

Spin Waves in Ferromagnetic and Antiferromagnetic Materials . . . . . F. KEFFER, H. KAPLAN, AND Y. YAFET 250

Čerenkov Radiation in a Dispersive Medium . . . . . H. MOTZ AND L. I. SCHIFF 258

Behavior of Magnetic Materials . . . . . R. M. BOZORTH 260

Rocking Experiment with Two Degrees of Freedom . . . . . JOHN SATTERLY 267

Geometrical Solution for a Nuclear Moment in a Magnetic Field . . ROALD K. WANGSNESS 274

Physics and Physiology in Diving Decompression . . . . . HILBERT SCHENCK, JR. 277

Calculus of Measures and Its Application to the Relation Between Rationalized and Unrationalized Electromagnetic Systems . . . . . J. R. BARKER AND R. O. DAVIES 281

Longitudinal Vibrations of a Vertical Column by the Method of Laplace Transform . L. H. HALL 287

Proposed Procedure for Selecting and Using Symbols for Physical Units . . DUANE ROLLER 293

Teaching a Philosophy of Experimentation in a Course in Electrical Measurements . . . . . FRANCIS T. WORRELL 296

Experimental Reactor Physics Course at the Oak Ridge School of Reactor Technology . . . . . E. D. KLEMA, R. J. STEPHENSON, AND S. TAYLOR 300

Surface Energy, A Mode for Energy Absorption During Impact . . . . . JOHN S. RINEHART 305

Undergraduate Experiment for the Determination of Alpha-Particle Range . . . . . WALTER C. MICHELS AND EVA WIENER 307

Relativistic Rocket Theory . . . . . W. L. BADE 310

Notes and Discussion:

Effect of Absorption by the Material of the Prism on Its Resolving Power . MAHENDRA SINGH SODHA 313

Determination of the Coefficient of Surface Tension by the Bubble-Length Method . . . . . JOHN C. SLONCZEWSKI AND RALPH HELLER 313

Equivalent Networks of Electron Tubes . . . . . SAUL ARONOW 314

Letters to the Editor:

Extension of the Conical Pendulum Problem and its Demonstration . . . JULIUS SUMNER MILLER 315

Ultrasonic Interferometer as a Screw Calibrating Instrument . . . . . ROMARD BARTHEL 316

Concerning the Electric Charge of a Moving Vehicle . . . . . JULIUS SUMNER MILLER 316

Demonstration of Subjective Harmonics . . . . . CHARLES WILLIAMSON 316

Trigonometry before Physics? . . . . . W. A. MURTAUGH 317

Announcements and News:

Book Reviews 318

Wisconsin Section 249

Indiana Section 257

Practical Aids for Physics Teachers 266

Tape Recordings of Important Speeches 276

To Secretaries of Regional Sections of AAPT 286

Summer Meeting of AAPT 309

Symposium on Molecular Structure 322

Recent Meetings:

Illinois Section 321